Pro-p Groups of Finite Width

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1 Introduction

The theory of pro-p groups of finite width is still in its infancy. So much so that even the definition of finite width is yet to be agreed upon. The following definition is pioneered in the recently published book [9] which concentrates on linear pro-p groups of finite width. The bulk of this interesting book is concerned with detailed computations of the lower central series of Sylow pro-p subgroups of various classical linear groups over local fields.

Definition 1 *A pro-p group G is said to be of finite width if the width*

$$w(G) := \sup_n \log_p(|\gamma_n(G)/\gamma_{n+1}(G)|)$$

is finite.

The point of this definition is to generalise the concept of finite coclass. Recall, a finite p-group H is said to have coclass $n-c$ when $|H| = p^n$ and the nilpotency class of H is c. A pro-p group has coclass $n-c$ if it is the inverse limit of finite p-groups of coclass $n-c$. Coclass has proved to be a very useful invariant when considering both finite p- and pro-p groups, [10] and [11]. We comment that for pro-p groups of finite coclass $|\gamma_n(G)/\gamma_{n+1}(G)| = p$

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for sufficiently large n.

Unlike pro-p groups of finite coclass, pro-p groups of finite width are not necessarily p-adic analytic. They need not even be linear, for example both the Nottingham group [2] and the pro-2 completion of the Grigorchuk group [6] are of finite width and neither p-adic analytic nor linear. Another interesting class of groups which are of finite width are the $\mathbb{F}_p[[t]]$-perfect groups, see [12, p.321], where $\mathbb{F}_p$ is the field of order p. It is unknown whether such groups are linear. In Section 2 it is shown that deciding whether a p-adic analytic pro-p group has finite width is simple. This does not seem to be true of pro-p groups in general.

The following definition of finite width has also been suggested [4, Definition 22], we call it finite central width.

Definition 2 *A pro-p group G has finite central width $w_c(G)$ if*

$$w_c(G) := \sup |H/K|$$

is finite, where H and K range over all central sections of G.

The Nottingham group, for odd p, has finite central width. Clearly finite central width implies finite width. The definition above could be weakened by asking only for a bound on central sections H/K with K open or equivalently H open, this would still imply finite width. One of the difficulties of the definition of finite central width is that it seems hard to check whether the property holds.

A. Mann and D. Segal have also introduced a definition of finite width [13, p.190], which bounds the *rank* of open central sections. Their motivation is to generalise the concept of Polynomial Subgroup Growth.

The following definitions were also introduced in [9, (I.1)], although originally the definition of finite average width was restricted to pro-p groups of finite width.

Definition 3 *A pro-p group G is said to be of finite average width if the average width*

$$w_a(G) := \lim_{n\to\infty} \frac{\log_p(|G : \gamma_{n+1}(G)|)}{n}$$

is finite.

Definition 4 *A pro-p group G is said to be of finite upper average width if the upper average width*

$$\overline{w}_a(G) := \limsup_{n\to\infty} \frac{\log_p(|G : \gamma_{n+1}(G)|)}{n}$$

is finite.

The pro-2 completion of the Grigorchuk group is an example of a pro-2 group of finite width but not of finite average width [6].

The aim of this article is to explore some of these definitions. We begin by considering p-adic analytic pro-p groups of finite width. We prove the following result, which although has a simple proof seems to previously have been missed.

Theorem A *Let G be a p-adic analytic pro-p-group. Then G has finite width if and only if $G/\gamma_2(G)$ is finite.*

By Theorem A if G is a p-adic analytic pro-p group the following is true:

$$w(G) \text{ is finite} \Leftrightarrow \overline{w}_a(G) \text{ is finite.}$$

Theorem A yields a positive answer to [9, Chapter XIV, Question (a)(i)] for the case when G is p-adic analytic, see Section 2. We also have

Corollary *Suppose $N \trianglelefteq G$ and that both G/N and N are p-adic analytic pro-p groups of finite width. Then, G is also a p-adic analytic pro-p group of finite width.*

This follows from Theorem A and [5, Theorem 10.8] since

$$|G : G'| = |G : G'N||G'N : G'| = |G : G'N||N : N \cap G'| \leq |G : G'N||N : N'|.$$

It is not known whether a finite extension of a pro-p group of finite width is always of finite width. However, finite (upper) average width and finite central width are extension closed, see Proposition 5 in Section 3.

Also in Section 3 there is a discussion of obliquity. In particular we give a condition on the *open* normal subgroups of a pro-p group of finite width which forces the pro-p group to be just infinite. Recall:

Definition 5 *A pro-p group G is just infinite if it has no non-trivial closed normal subgroup of infinite index.*

Finally in Section 4 we concentrate on which properties of pro-p groups are compatible with being of finite width. In particular we prove the following result.

Theorem B *A pro-p group cannot have all three of the following properties: powerful, soluble, finite width.*

Note that pro-p groups exist with any two of the stated properties.

Our last comments are related to the existence of torsion-free virtually abelian pro-p groups of finite width. Given a discrete group, say Γ which is torsion-free and virtually abelian with finite quotient a p-group, then Γ can be embedded in the pro-p completion which will also be torsion-free and virtually abelian. The existence of torsion-free virtually abelian discrete groups is related to the existence of a flat compact Riemannian manifold X on which Γ acts as the fundamental group [1]. The requirement that Γ/Γ' be finite is equivalent to asking that the first Betti number, $b_1(X)$, is 0 [7]. Using this and the results of [14] we give a description of the finite p-groups which occur as point groups of torsion-free virtually abelian pro-p group of finite width.

Throughout this paper a subgroup of a topological group is supposed to be closed.

2 p-adic analytic

For ease we introduce the following definition.

Definition 6 *A pro-p group G has weak finite width if* $|\gamma_n(G)/\gamma_{n+1}(G)|$ *is finite for all* $n \geq 1$.

Note that a pro-p group of weak finite width is finitely generated. Consider the ring $\Lambda = \mathbb{Z}_p[[t]]$, where $\mathbb{Z}_p$ is the ring of p-adic integers. This is a complete local Noetherian ring with maximal ideal $M = \langle p, t \rangle$. Let G be a Λ-perfect group, of dimension d. Then $\log_p |\gamma_n(G) : \gamma_{n+1}(G)| = d \cdot \dim(M^n/M^{n+1})$ by [12, Theorem 3.5(1)]. Thus we have examples of pro-p groups of weak finite width but not of finite width. Notice that these groups do not satisfy Mann & Segal's definition of finite width and are not p-adic analytic.

Lemma 1 *Let G be a finitely generated pro-p group and H an open subgroup of G. If H' is open in H then H has weak finite width.*

Proof. Since H/H' is finite it has finite exponent. So each term of the lower central series $\gamma_n(H)/\gamma_{n+1}(H)$ has finite exponent, see [8, III.2.14]. We now prove the result by induction. Assume that $|\gamma_r(H)/\gamma_{r+1}(H)|$ is finite for all $r \leq n$. So $\gamma_{n+1}(H)$ is open and so finitely generated [5, 1.7]. So $\gamma_{n+1}(H)/\gamma_{n+2}(H)$ is abelian, finitely generated and of finite exponent. So is finite.□

A pro-p group G is p-adic analytic if and only if each open subgroup has a bounded number of generators [5, 9.35]. This is crucial for the following lemma.

Lemma 2 *Let G be a p-adic analytic pro-p group and H an open subgroup of G. If H' is open in H then H has finite width.*

Proof. The proof is as in Lemma 1 but this time the number of generators of $\gamma_n(H)$ is bounded.□

This leads to Theorem A.

Remark 1 The following known fact [9, p.4] follows immediately from Theorem A. A just-infinite p-adic analytic pro-p group is of finite width or isomorphic to $\mathbb{Z}_p$.

The following question was raised in [9, Chapter XIV, (a)(i)]. Recall, a hereditarily just infinite pro-p group is one in which every open subgroup is just infinite, equivalently every subnormal subgroup is open.

Question. Assume G is an insoluble hereditarily just infinite pro-p group of finite width. Is every open subgroup H of G of finite width?

Note that the properties of G imply that H' is open in H thus Lemma 2 affirmatively answers this question for the case when G is p-adic analytic.

Assume now that G is only insoluble and just infinite. Let H be an open subgroup of G and consider $\mathrm{core}(H) = \bigcap gHg^{-1}$. This is an open normal subgroup of G and so $\mathrm{core}(H)'$ is open. Now it is easy to see that H' is also open in G. Perhaps the question should be modified to ask

Question. Assume G is an insoluble just infinite pro-p group of finite width. Is every open subgroup H of G of finite width?

Again by Lemma 2 the answer is yes if G is p-adic analytic.

We now prove generalisations of [9, (II.1)] and [9, (II.4)]. The proofs follow closely the proofs given in [9], with the benefit of clearer inductive steps.

Proposition 1 *Let G be a pro-p group of finite (upper) average width and H an open subgroup of G. If H' is open in H then H has finite (upper) average width.*

Proof. The proof is by induction on $|G : H|$. We begin by showing that the result is true when $|G : H| = p$.

Since H' is open we have that H/H' has finite exponent say p^a. Consider the group ring $R = (\mathbb{Z}/p^a\mathbb{Z})C_p$, where C_p denotes the cyclic group of order p. Now for some k the k^{th} power of the augmentation ideal of R will vanish. Hence $[\gamma_n(H),_k G] \leq \gamma_{n+1}(H)$ for any $n \in \mathbb{N}$. We now use induction on m to show that $\gamma_{km}(G) \leq \gamma_m(H)$. As G/H is abelian, $\gamma_2(G) \leq \gamma_1(H)$. Then

$$\gamma_{k(m+1)}(G) = [\gamma_{km}(G),\, _kG] \leq [\gamma_m(H),\, _kG] \leq \gamma_{m+1}(H).$$

So

$$|H : \gamma_m(H)| \leq |G : \gamma_{mk}(G)|.$$

The result now follows for any subgroup of index p. Let H be any open subgroup with H' open, then there exists K with $|G : K| = p$ and $H \subseteq K$ and K' open. Now K satisfies the hypothesis of the proposition and so by induction so does H. □

Proposition 2 *Let G be a pro-p group of finite central width. Let H be a normal open subgroup of G with H' open in H. Then H has finite width.*

Proof. The proof is similar to the proof of the previous proposition. Let H/H' have exponent p^a and let A be the finite p-group G/H. Let $R = (\mathbb{Z}/p^a\mathbb{Z})A$. There exists a natural number k so that the k^{th} power of the augmentation ideal of R vanishes. As above $[\gamma_n(H),\, _kG] \leq \gamma_{n+1}(H)$. Now $[\gamma_n(H),\, _rG]/[\gamma_n(H),\, _{r+1}G]$ is a central section of G and so has order bounded by $p^{w_c(G)}$. But then $|\gamma_n(H) : \gamma_{n+1}(H)| \leq p^{kw_c(G)}$. Thus H has finite width.□

Remark 2 So the properties of finite (upper) average width are inherited by open subgroups H with H' open. It is still unknown whether the properties of having finite width or finite central width are inherited similarly.

3 Obliquity & finite central width

The following lemma is useful.

Lemma 3 *Let G be a pro-p group and $N \trianglelefteq G$. If $\gamma_n(G) \le N\gamma_{n+1}(G)$ then $\gamma_n(G) \le N$.*

Proof. We prove by induction that $\gamma_n(G) \le N\gamma_{n+m}(G)$ for all $m \in \mathbb{N}$ then the result follows since $\bigcap \gamma_{n+m}(G) = 1$. Suppose $\gamma_n(G) \le N\gamma_{n+m}(G)$ then

$$\gamma_{n+1}(G) \le [N\gamma_{n+m}(G), G] = [N, G][\gamma_{n+m}(G), G] = [N, G]\gamma_{n+m+1}(G).$$

So $\gamma_n(G) \le N\gamma_{n+1}(G) \le N\gamma_{n+m+1}(G)$ and the inductive step is complete.□

The following result although simple is surprising, since it gives a condition on the *open* normal subgroups of a pro-p group of weak finite width, which implies that the group is just infinite.

Proposition 3 *Let G be a pro-p group of weak finite width. Suppose there exists $t \in \mathbb{N}$ such that if N is an open normal subgroup of G then for some n,*

$$\gamma_{n+t}(G) \subseteq N \subseteq \gamma_n(G).$$

Then G is just infinite.

Proof. Suppose G is not just infinite and that M is a non-trivial closed normal subgroup of G of infinite index. Then there exists maximal n such that $M \subseteq \gamma_n(G)$. Since $M\gamma_{n+t+1}(G)$ is an open normal subgroup of G it follows that $M\gamma_{n+t+1}(G) \supseteq \gamma_{n+t}(G)$. By Lemma 3 it follows that $M \supseteq \gamma_{n+t}(G)$ and consequently that M is open, a contradiction.□

It was stated in [9, p.10] that any pro-p group of finite obliquity has the property stated in the above proposition. We show, in Proposition 4, that these two conditions are equivalent. Recall the definition of obliquity [9, I.5]:

Definition 7 *Let G be a pro-p group of finite width. Put*

$$\mu_n(G) = \gamma_{n+1}(G) \cap \bigcap \{N \lhd G : N \not\le \gamma_{n+1}(G)\}.$$

The obliquity of G is defined to be

$$o(G) = \sup_n \log_p |\gamma_{n+1}(G) : \mu_n(G)| \in \mathbb{N} \cup \infty.$$

In [9, III(d)] it is shown that just infinite, insoluble p-adic analytic pro-p groups have finite obliquity. For p odd the Nottingham group has obliquity 0.

Proposition 4 *Let G be a pro-p group of finite width. Then the following two conditions are equivalent.*
(i) G has finite obliquity,
(ii) there exists an integer t so that if N is an open normal subgroup of G then for some n,

$$\gamma_{n+t}(G) \subseteq N \subseteq \gamma_n(G).$$

Further they imply
(iii) if H/K is a central section of G with K open then $|H : K|$ is bounded.

Proof. (i)⇒(ii). Let G have finite obliquity p^s. Let N be an open normal subgroup of G. Assume that $N \subseteq \gamma_n(G)$ but $N \not\subseteq \gamma_{n+1}(G)$. Then $|\gamma_{n+1}(G) : (\gamma_{n+1}(G) \cap N)| \leq p^s$. But then $N \cap \gamma_{n+1}(G) \supseteq \gamma_{n+s+1}(G)$. Thus (ii) holds with $t = s + 1$.

(ii)⇒(i). Choose some m and let N be a normal subgroup of G with $N \not\subseteq \gamma_{m+1}(G)$. N is open by Proposition 3, therefore by (ii) $N \supseteq \gamma_{m+t}(G)$. So $\mu_m(G) \supseteq \gamma_{m+t}(G)$ and $|\gamma_{m+1}(G) : \mu_m(G)| \leq p^{(t-1)w}$, where $w = w(G)$.

Finally we show that (ii)⇒(iii). Let H/K be a central section of G with K open and let G satisfy (ii). Then $\gamma_m(G) \supseteq H \supseteq \gamma_{m+t}(G)$ for some m. Hence $K \supseteq [H, G] \supseteq \gamma_{m+t+1}(G)$. So $|H : K| \leq |\gamma_m(G) : \gamma_{m+t+1}(G)|$ and so (iii) holds. □

Note that (i) is a condition on all normal subgroups of G, whereas (ii) just considers open normal subgroups. That (iii) is strictly weaker than (ii) can be seen by considering the direct product $H \times K$ of two pro-p groups of finite central width. Let U be an open normal subgroup of H then $U \times \gamma_n(K)$ is an open normal subgroup of $H \times K$ for all n, hence (ii) is not satisfied. However, by Proposition 5 part (iii) $H \times K$ is of finite central width.

Proposition 5 *Let G be a pro-p group and N be a normal subgroup of G.*

(i) If both G/N and N have finite (upper) average width then G has finite (upper) average width.

(ii) If G/N has finite width and N has finite central width then G has finite width.

(iii) If both G/N and N have finite central width then G has finite central width.

Proof. That G has finite (upper) average width follows from

$$\begin{aligned} |G:\gamma_n(G)| &= |G:\gamma_n(G)N||\gamma_n(G)N:\gamma_n(G)| \\ &= |G:\gamma_n(G)N||N:N\cap\gamma_n(G)| \\ &\leq |G:\gamma_n(G)N||N:\gamma_n(N)|. \end{aligned}$$

For (ii) consider the following.

$$\begin{aligned} |\gamma_{n-1}(G):\gamma_n(G)| &= |\gamma_{n-1}(G):\gamma_n(G)N\cap\gamma_{n-1}(G)||\gamma_n(G)N\cap\gamma_{n-1}(G):\gamma_n(G)| \\ &= |\gamma_{n-1}(G)N:\gamma_n(G)N||\gamma_n(G)(N\cap\gamma_{n-1}(G)):\gamma_n(G)| \\ &= |\gamma_{n-1}(G)N:\gamma_n(G)N||N\cap\gamma_{n-1}(G):N\cap\gamma_n(G)|. \end{aligned}$$

As G/N is of finite width $|\gamma_{n-1}(G)N : \gamma_n(G)N|$ is bounded. Also, since central sections of N are bounded $|N\cap\gamma_{n-1}(G) : N\cap\gamma_n(G)|$ is bounded.

The proof of (iii) is similar.□

4 Compatibility

For a pro-p group G we denote by G_m the m^{th} term of the Frattini series (also known as the lower p-series). In a finitely-generated pro-p group this series consists of open subgroups [5, 1.16(iii)].

Proposition 6 *Let G be a powerful pro-p group of finite width. Then G_m has finite width for all m.*

Proof. First note that G is p-adic analytic [5, 9.34], and therefore so is G_m for all m. Assume that some G_m does not have finite width. Then by Lemma 2 $[G_m, G_m]$ must have infinite index in G_m. We factor out by $[G_m, G_m]$.

The quotient group will still be a powerful pro-p group of finite width. Since it is powerful the elements of finite order form a characteristic finite subgroup, T [5, 4.20]. We now factor by T to get $\bar{G}$, a powerful, pro-p group of finite width which is also torsion free. Now consider $\bar{G}_m$. This is an open abelian powerfully embedded subgroup of $\bar{G}$, from [5, 3.6]. By the proof of [5, 4.20] we can see that $\bar{G}_m$ must be central. This contradicts the fact that $\bar{G}$ is of finite width.□

We are now ready to prove Theorem B.

Theorem B *A pro-p group cannot have all three of the following properties:*

powerful, soluble, finite width.

Proof. Suppose G is a pro-p group with all three of these properties. By solubility there exists minimal n such that $G^{(n)}$ has a derived group of infinite index, i.e. $|G^{(n)} : \gamma_2(G^{(n)})|$ is not finite. Since $G^{(n)}$ is open there exists minimal m such that $G^{(n)} \geq G_m$. It follows that $[G_m, G_m]$ is of infinite index in G_m. This contradicts the preceding theorem.□

Remark 3 By Theorems A and B non-abelian, soluble, just infinite pro-p groups are not powerful. (A soluble, just infinite pro-p group is p-adic analytic by [15, 8.1].)

Note that any p-adic analytic pro-p group has an open powerful subgroup [5, 9.36].

Corollary 1 *Let G be a p-adic analytic soluble pro-p group. Let H be an open normal powerful subgroup of G. Then $H/[H, H]$ is infinite.*

Proof. Since H is soluble and powerful we have from Theorem B that H does not have finite width. So H' is not open by Lemma 2.□

We now turn our attention to the question of the existence of a pro-p group, G, which is virtually abelian, torsion-free and of finite width. A simple argument shows that such a group cannot be just infinite. Note that the group G has finite width if and only if G/G' is a finite group by Theorem A.

In such a group G there is an abelian normal subgroup, say A with $P = G/A$ a finite p-group, we may assume that A is self-centralising. A is a free $\mathbb{Z}_p$-module on which P acts so that A is a $\mathbb{Z}_p P$-module. This is to say that we are looking for a p-adic space group which is torsion-free with P as the point group.

Assume that there is a torsion-free crystallographic space group Γ with point group P and translation group M. W. Plesken [14, Remark II.1] pointed out that the inverse limit of the system given by normal subgroups of Γ given by $p^a M$ will give rise to a torsion-free p-adic space group with point group P. This p-adic space group will have a finite abelianisation if and only if Γ does. Thus we are interested in the existence of Γ.

In [1, Theorem 1] it is shown that a group Γ is the fundamental group of a flat compact Riemannian manifold if and only if it is a torsion-free virtually abelian group. They also show how to construct such a group with any given finite group as quotient. However the construction always has Γ/Γ' infinite. H. Hiller and C-H. Sah [7, 1.4] prove that a torsion-free virtually abelian group has Γ/Γ' finite if and only if the first Betti number of the space on which it acts is zero if and only if $Z(\Gamma) = 1$. They call finite groups which can occur as the finite quotient, Γ/A where A is the torsion-free abelian group, 'primitive'. They show [7, 3.6] that a p-group is 'primitive' if and only if it is not cyclic. Thus for any non-cyclic p-group P there is a torsion-free pro-p group of finite width with point group P. There is a nice alternative approach by Cliff and Weiss [3].

We now wish to consider whether given a torsion-free virtually abelian pro-p group G, does it arise from a discrete group defined as above. This is confirmed by Plesken [14, IV]. When the point group is a p-group much of the argument in that section is simpler.

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