Trivializing the Hrushovski constructions

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EHUD HRUSHOVSKI: (1988) Counterexamples to two of the most significant conjectures in model theory.

QUESTION: Are the counterexamples just very clever pathologies, or do they have connections with other parts of mathematics?

THIS TALK:

- Model-theoretic background
- Zilber’s conjecture
- Hrushovski constructions
- Random graphs (Shelah, Spencer; Baldwin)
- New way of looking at the constructions (DE)
1. Model theory

The *formulas* of a first-order language \( L \) are certain finite strings of the symbols:

\[ \forall \exists \to \to \wedge \vee \) ( , \( x_1 x_2 \ldots y_1 y_2 \ldots \)

and

(2) Various symbols (including \( = \)) used to denote relations and functions.

What you take for (2) depends on what sort of structure you want the formulas to talk about.

**EXAMPLES**: (i) Graphs: \( = \) and a 2-ary relation \( R \) for adjacency.

(ii) Rings: \( = \) and \( +, \cdot \) (2-ary functions), \( 0, 1 \) (constants).

(iii) \( K \)-vector spaces: \( =, +, 0 \), and for each \( \alpha \in K \) a 1-ary function symbol to denote scalar multiplication by \( \alpha \).

**\( L \)-FORMULAS**: Usual mathematical shorthand: variables can only range over the *elements* of a structure.

**NOTATION**: (i) \( M \models \phi \) the formula \( \phi \) is true in the structure \( M \).

(ii) If \( \phi(x_1, \ldots, x_n, y_1, \ldots, y_m) \) is a formula with free variables amongst \( x_1, \ldots, x_n, y_1, \ldots, y_m \) and \( \bar{a} = (a_1, \ldots, a_m) \in M^m \), let

\[ \phi[M, \bar{a}] = \{ (b_1, \ldots, b_n) \in M^n : M \models \phi(b_1, \ldots, b_n, \bar{a}) \} \]

This is a *definable subset of* \( M^n \) (using parameters \( a_1, \ldots, a_m \)).
GENERAL PHILOSOPHY: Fix a language $L$ and:

(I) Compare $L$-structures by looking at their $L$-theories

$$Th(M) = \{ \phi : \phi \text{ closed and } M \models \phi \}.$$ 

(II) For a given $L$-structure $M$, think about its collection of definable subsets.

EXAMPLES FOR (I): What properties can be expressed by first-order formulas?

**Graphs:**
- Triangle free (YES)
- Diameter $\leq d$ (YES)
- Connected (NO)

**Rings:**
- Integral domain (YES)
- Bézout (YES)
- Principal ideal domain (NO)
2. Zilber’s Conjecture.

**Definition:** An infinite $L$-structure $M$ is *strongly minimal* if for every $L$-formula $\phi(x, \bar{y})$ there exists $k \in \mathbb{N}$ such that for all $\bar{a}$, either 
\[ \{ b \in M : M \models \phi(b, \bar{a}) \} \] or its complement has size $\leq k$.

From the viewpoint of (II), these are the ‘simplest’ structures.

**Examples of strongly minimal structures:**
1. $M$ is a ‘pure set’ (the language $L$ has $=$, but no other relation or function symbols).
2. $M$ is a $K$-vector space (where $K$ is a division ring and the language is as described before).
3. $M$ is an algebraically closed field (the language is the language for rings).

**Zilber’s Conjecture:** These are essentially the only examples of strongly minimal structures.

Early 1980’s. **Theorem** (Zilber et al.): The conjecture is true for $\omega$-categorical structures.

1988. Without any further hypotheses, the conjecture is false (Hrushovski).

Early 1990’s. Under additional hypotheses (Zariski structure) the conjecture is true (Hrushovski, Zilber).

1990’s - date. New idea of Zilber: Realise the counterexamples in ‘classical’ mathematics using complex analytic functions.

Work of Zilber, Wilkie, Koiran, Peatfield....
2003. Zilber: Connections between the construction and non-commutative geometry, string theory...
3. The construction

Describe the simplest form of the construction.
Work with graphs (so $L$ has $=$ and a 2-ary relation symbol $R$).
Fix a real parameter $\alpha$ with $0 < \alpha < 1$.

**Definition:**

(1) If $A$ is a finite graph define the *predimension* of $A$ to be

$$\delta(A) = |A| - \alpha e(A)$$

where $e$ denotes the number of edges in $A$.

(2) If $A$ is a subgraph of the finite graph $B$ write

$$A \subseteq B$$

to mean

$$\delta(A) \leq \delta(B')$$

for all $B'$ with $A \subseteq B' \subseteq B$.

(Pronounced: $A$ is a self-sufficient subgraph of $B$.)

**Properties:**

(1) If $A \subseteq B$ and $X \subseteq B$, then $A \cap X \subseteq X$.

(2) If $A \subseteq B \subseteq C$, then $A \subseteq C$.

(3) If $A_1, A_2 \subseteq B$, then $A_1 \cap A_2 \subseteq B$.

(4) If $X \subseteq B$, there is a unique smallest $A \subseteq B$ with $X \subseteq A$. Call this the *closure* of $X$ in $B$, and denote it by $\text{cl}_B(X)$. 


Denote by $C$ the class of finite graphs $A$ which satisfy

$$\emptyset \leq A$$

i.e. for all $X \subseteq A$, we have $|X| - \alpha e(A) \geq 0$. (Another way: average valency of $X$ is $\leq 2/\alpha$.)

**Strong Amalgamation Lemma:** Suppose $B, C \in C$ and $A$ is a subgraph of both $B$ and $C$, and $A \leq C$. Let $E$ be the disjoint union of $B$ and $C$ over $A$. Then $E \in C$ and $B \leq E$.

Using this, we can ‘glue’ the graphs in $C$ together to obtain:

**Theorem:** There exists a countably infinite graph $M = M_\alpha$ satisfying the following properties:

(G1): $M$ is the union of a chain of finite subgraphs $A_1 \leq A_2 \leq A_3 \leq \cdots$ all in $C$.

(G2): If $A \leq M$ is finite and $A \leq B \in C$, then there is an embedding $f : B \rightarrow M$ which is the identity on $A$ and has $f(B) \leq M$.

Moreover, $M$ is uniquely determined up to isomorphism by these two properties and if $h : B_1 \rightarrow B_2$ is an isomorphism between finite closed subgraphs of $M$, then $h$ can be extended to an automorphism of $M$. \(\square\)

**Theorem:** (Hrushovski; Wagner; Baldwin, Shi) If $0 < \alpha < 1$ then $M_\alpha$ is stable (and not 1-based). If $\alpha$ is rational, then $M_\alpha$ is $\omega$-stable, of infinite Morley rank. \(\square\)
4. Irrational $\alpha$, random graphs

S. Shelah, J. Spencer, (JAMS, 1988): Fix $\alpha$ irrational with $0 < \alpha < 1$. For $n \in \mathbb{N}$, consider choosing a graph on $n$ vertices by randomly choosing each pair of vertices to be an edge, with probability $1/n^\alpha$. If $\phi$ is a closed $L$-formula, let

$$P(\phi, \alpha; n)$$

be the probability that the randomly chosen graph has the property expressed by $\phi$. Consider what happens as $n \to \infty$:

**Theorem**: (Zero-one law) For each such $\phi$, either $P(\phi, \alpha; n) \to 0$ as $n \to \infty$, or $P(\phi, \alpha; n) \to 1$ as $n \to \infty$.

Later on, Baldwin and Shelah made the connection:

**Theorem**: For all closed $L$-formulas $\phi$:

$$P(\phi, \alpha; n) \to 1 \text{ as } n \to \infty \iff M_\alpha \models \phi.$$

**Remarks**: (1) Compare with the classic result of Fagin, Glebskii et al.. If we choose the edges with probability $\frac{1}{2}$, then we again have a zero-one law, but this time the limit theory is that of the Random Graph.

(2) If $\beta$ is rational and $0 < \beta < 1$ then as $\alpha \to \beta^-$ (and $\alpha$ irrational), then $Th(M_\alpha) \to Th(M_\beta)$. 

5. α rational; directed graphs

Directed graphs: Let $\mathcal{D}$ be the class of finite directed graphs $D$ with all vertices having $\leq 2$ out-vertices. If $C \subseteq D$, write $C \subseteq D$ to mean that out-vertices of elements of $C$ are contained in $C$ (say that $C$ is closed in $D$).

Easy Lemma: (1) If $C \subseteq D$ and $X \subseteq D$ then $C \cap X \subseteq X$.
(2) If $C \subseteq D \subseteq E$ then $C \subseteq E$.
(3) (Strong Amalgamation) Suppose $D, E \in \mathcal{D}$ and $C$ is a sub-digraph of both $D$ and $E$ and $C \subseteq E$. Let $F$ be the disjoint union of $D$ and $E$ over $C$. Then $F \in \mathcal{D}$ and $D \subseteq F$. □

Using this we have:

Proposition: There exists a countably infinite digraph $N$ satisfying the following properties:

(D1): $N$ is the union of a chain of finite subgraphs $C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots$ all in $\mathcal{D}$.

(D2): If $C \subseteq N$ is finite and $C \subseteq D \in \mathcal{D}$, then there is an embedding $f : D \rightarrow N$ which is the identity on $C$ and has $f(D) \subseteq N$.

Moreover, $N$ is uniquely determined up to isomorphism by these two properties and is $\subseteq$-homogeneous. □

Proposition: $N$ is stable, trivial and 1-based. □

... So $N$ is rather a dull structure.
.... or is it?

Fix \( \alpha = \frac{1}{2} \). Work with \( \delta(A) = 2|A| - e(A) \).

So \( C = \{A : \delta(X) \geq 0 \text{ for all } X \subseteq A\} \) and \( M = M_{1/2} \).

**Theorem:** Forget the directions on the edges in \( N \). The resulting graph is \( M_{1/2} \).

The following answers a question of Bruno Poizat from 1991.

**Corollary:** There is a stable, trivial, 1-based structure with a reduct which is neither trivial, nor 1-based.

**Definition:** Suppose \( A \) is a finite graph. A \( \mathcal{D} \)-orientation of \( A \) is a directed graph \( A^+ \in \mathcal{D} \) with the same vertex set as \( A \) and such that if we forget the direction on the edges, we obtain \( A \).

The theorem is a fairly straightforward corollary of the following two lemmas:

**Lemma 1:** (1) Suppose \( B \) is a finite graph. Then

\[
B \in C \iff B \text{ has a } \mathcal{D} \text{-orientation.}
\]

(2) If \( B \in C \) and \( A \subseteq B \), then \( A \leq B \) iff there is a \( \mathcal{D} \)-orientation of \( B \) in which \( A \) is closed.

**Lemma 2:** If \( A \leq B \in C \) then any \( \mathcal{D} \)-orientation of \( A \) extends to a \( \mathcal{D} \)-orientation of \( B \).