Model-theoretic constructions for $\omega$-categorical structures

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**ω-categoricity**

**NOTATION:** $L$ a first-order language; $M$ a countably infinite $L$-structure.

**DEFINITION:** $M$ is **ω-categorical** if every countable model of $Th(M)$ is isomorphic to $M$.

**FACTS:** (Engeler, Ryll-Nardzewski, Svenonius) Let $G = \text{Aut}(M)$. Then $M$ is ω-categorical iff $G$ has finitely many orbits on $M^n$ (for all $n \in \mathbb{N}$).

Orbits: $\{(ga_1, \ldots, ga_n) : g \in G\}$ for $(a_1, \ldots, a_n) \in M^n$.

If $M$ is ω-categorical then:

$G$-orbits on $M^n$ correspond to complete $n$-types over $\emptyset$.

**NOTE:** If $M$ is ω-categorical, then it is **locally finite**: any finitely generated substructure is finite.
Constructions of $\omega$-categorical structures

1. **Examples in nature:**

   - Pure set $\Omega$ (automorphism group $\text{Sym}(\Omega)$)
   - $(\mathbb{Q}, \leq)$ (Cantor’s theorem)
   - Vector spaces $V(\omega, q)$ over finite fields
   - ...

2. **New structures from old ones:**

   - Finite products; covers.
   - Any structure interpretable in a $\omega$-categorical structure is $\omega$-categorical. For example:
     - $n$-sets from a pure set $([\Omega]^n$ with $\text{Sym}(\Omega)$ as automorphism group)
     - Reducts (mysterious, but interesting)

3. **Boolean powers:**

   - Important in, for example, $\omega$-categorical groups.

4. **Amalgamation methods:**

   - The main focus of this talk.
Amalgamation: the basic Fraïssé construction

A class $C$ of finite $L$-structures is an amalgamation class if:

- $C$ has countably many isomorphism types
- $C$ is closed under substructures
- (Joint embedding) Any two structures in $C$ can be embedded in a third
- (Amalgamation) If $A, B_1, B_2 \in C$ and $f_i : A \to B_i$ are embeddings there exists $C \in C$ and embeddings $g_i : B_i \to C$ with $g_1 \circ f_1 = g_2 \circ f_2$.

Given this, there exists a chain of structures in $C$:

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_i \subseteq \cdots$$

such that:

- Every structure in $C$ is isomorphic to a substructure of some $M_i$
- If $A$ is a substructure of $M_i$, and $B \in C$ and $f : A \to B$ is an embedding, then there exists $j \geq i$ and an embedding $g : B \to M_j$ such that $g \circ f$ is the identity on $A$.

Let $M = \bigcup_{i \in \mathbb{N}} M_i$. Then:

1. $M$ is countable and locally finite
2. $\text{Age}(M) = C$
3. If $A \subseteq M$ is a finite substructure, $B \in C$ and $f : A \to B$ is an embedding, then there exists and embedding $g : B \to M$ with $g \circ f$ the identity on $A$.

Moreover, using a back-and-forth argument:

- Properties 1, 2, 3 determine $M$ up to isomorphism
- ((Ultra-)Homogeneity) Any isomorphism between between finite substructures of $M$ extends to an automorphism of $M$

Refer to $M$ as the Fraïssé limit or generic structure of the amalgamation class $C$. 
**Theorem:** (R. Fraïssé) A (locally finite) countable structure $M$ is homogeneous iff $\text{Age}(M)$ is an amalgamation class.

**Notes:**
1. Homogeneous structure $M$ is $\omega$-categorical iff it is locally finite and for each $n \in \mathbb{N}$ there are finitely many isomorphism types of $n$-generator substructures of $M$.
2. An $\omega$-categorical structure is homogeneous (in this sense) iff it has QE.

**Examples of Amalgamation Classes:**

1. Finite graphs (- Fraïssé limit is the random graph)
2. Finite graphs omitting the complete graph on $n$ vertices ($n$ fixed)
3. Finite digraphs
4. Finite digraphs omitting a given set of tournaments
5. Finite posets
6. Finite distributive lattices
7. Finite groups (- Fraïssé limit is Philip Hall’s countable universal locally finite group)

In Examples 1-4 we can take amalgamation to be *free amalgamation*. In all cases apart from 7, the limit is $\omega$-categorical.
Variations on the basic construction

**IDEA:** Work with a class $\mathcal{K}$ of finite $L$-structures and a notion:

$$A \sqsubseteq B$$

pronounced ‘$A$ is a nicely embedded substructure of $B$.’ Demand the amalgamation property only over *nicely embedded* substructures. More formally, work with $\sqsubseteq$-embeddings $f : A \to B$ - meaning $f(A) \sqsubseteq B$. We’ll *assume* that these embeddings include isomorphisms; are closed under composition (so $\sqsubseteq$ is transitive); and under restriction of the codomain.

Say that $(\mathcal{K}, \sqsubseteq)$ is an *amalgamation class* if:

- $\mathcal{K}$ is closed under $\sqsubseteq$-substructures
- $\mathcal{K}$ has countably many isomorphism types
- (Joint embedding) Any two elements of $\mathcal{K}$ can be $\sqsubseteq$-embedded in a third.
- ($\sqsubseteq$-Amalgamation) If $A, B_1, B_2 \in \mathcal{K}$ and $f_i : A \to B_i$ are $\sqsubseteq$-embeddings, there exist $C \in \mathcal{K}$ and $\sqsubseteq$-embeddings $g_i : B_i \to C$ with $g_1 \circ f_1 = g_2 \circ f_2$. 
**Theorem:** There is a structure $M$ satisfying:

1. $M$ is the union of a chain $M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$ of members of $\mathcal{K}$
2. Any member of $\mathcal{K}$ is isomorphic to a $\subseteq$-substructure of $M$
3. If $A \subseteq M$ is finite and $f : A \to B \in \mathcal{K}$ is a $\subseteq$-embedding there is a $\subseteq$-embedding $g : B \to M$ with $g \circ f$ the identity on $A$.

Moreover $M$ is uniquely determined by these properties and any isomorphism between finite $\subseteq$-substructures of $M$ extends to an automorphism of $M$.

**Notes:**

1. We will call $M$ here the *generic structure* for the class $(\mathcal{K}, \subseteq)$.

2. Suppose there are only finitely many isomorphism types of structures in $M$ of any finite size. Suppose also that there is a function $F : \mathbb{N} \to \mathbb{N}$ with the property that if $B \in \mathcal{K}$ and $X \subseteq B$ has size $\leq n$ then there exists $A \subseteq B$ containing $X$ and $|A| \leq F(n)$. Then $M$ is $\omega$-categorical.

**Example:** (Not $\omega$-categorical) Let $\mathcal{K}$ be the class of finite digraphs in which the number of edges coming out of any vertex is at most $2$. Write $A \subseteq B$ to mean that there are no edges coming out of $A$ (in $B$).

**(Puzzle:** Take the generic here and forget the direction on the edges. Describe the resulting graph.)
Hrushovski’s construction

Work with graphs.
Let \( \alpha \) be a fixed positive real number. If \( B \) is a finite graph define the **predimension** of \( B \) as:

\[
\delta(B) = |B| - \alpha e(B)
\]

where \( e(B) \) is the number of edges in \( B \). If \( A \subseteq B \) write

\[
A \preceq B \iff \delta(A) < \delta(B_1) \text{ whenever } A \subset B_1 \subseteq B.
\]

**NOTES:**
1. Compare with dimension in a vector space.
2. There is a related notion \( A \preceq^* B \): have \( \leq \) rather than \( < \).

**Lemma:**
1. If \( A \preceq B \preceq C \), then \( A \preceq C \).
2. If \( X \subseteq B \) and \( A \preceq B \), then \( A \cap X \subseteq X \).
3. If \( X \subseteq B \), then \( \bigcap\{ A : X \subseteq A \preceq B \} \subseteq B \).

Call the set in 3. the **closure** of \( X \) in \( B \).

**Example:** Take \( \alpha = 1/2 \). In each case \( B \) is the closure of the two points in \( X \):

![Diagram](image_url)
**Definition:** Let \( \mathcal{K}_0 \) consist of finite graphs \( A \) with \( \emptyset \leq A \): i.e. for every non-empty subgraph \( A_1 \) of \( A \) we have \( |A_1| - \alpha e(A_1) > 0 \).

**Lemma:** \((\mathcal{K}_0, \leq)\) is an amalgamation class.

**Proof.** Show that if \( A \leq B_1, B_2 \in \mathcal{K}_0 \) then the free amalgam \( E \) of \( B_1 \) and \( B_2 \) over \( A \) is in \( \mathcal{K}_0 \) and \( B_1, B_2 \leq E \). If \( F \subseteq E \) then \( F \) is the free amalgam over \( F \cap A \) of \( F \cap B_1 \) and \( F \cap B_2 \) and \( F \cap A \leq F \cap B_i \).

So the only calculation we really need is:

\[
\delta(E) = \delta(B_1) + \delta(B_2) - \delta(A) > \delta(B_1) > 0
\]

assuming we’re not in a trivial case where \( A = B_1 \) or \( A = B_2 \).

The generic for \((\mathcal{K}_0, \leq)\) is **not** \( \omega \)-categorical. The size of the closure of \( k \) points is not bounded by a function of \( k \).

**Idea… for obtaining \( \omega \)-categoricity:**

Take a continuous, increasing bijection \( F : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} \) with \( F(x) \to \infty \) as \( x \to \infty \). Let \( \mathcal{K}_F \) consist of all finite graphs \( B \) with

\[
\delta(A) \geq F(|A|)
\]

for all \( A \subseteq B \).

**Observation:** If \( X \subseteq B \in \mathcal{K}_F \) then the closure of \( X \) in \( B \) has size \( \leq F^{-1}(\delta(X)) \).

So if \((\mathcal{K}_F, \leq)\) has the amalgamation property, **then** it is an amalgamation class and the generic structure \( M_F \) is \( \omega \)-categorical.
How you choose $F$ to obtain the amalgamation property depends on the $\alpha$ used to define the predimension.

**EXAMPLES:**

1. (Rational $\alpha$; Hrushovski, 1988) Suppose $\delta(A) = 2|A| - e(A)$. Choose $F$ right-differentiable (e.g. piecewise linear), with right derivative $F'(x)$ non-increasing and $F'(x) \leq 1/x$.

2. (Irrational $\alpha$ of ‘infinite index’; Hrushovski, 1988) Choice of $F$ is more subtle.
Model-theoretic properties: $\omega$-categorical case

1. (E. Hrushovski, 1988) Take $\alpha$ an appropriate irrational and a suitable $F$. The generic $M_F$ is $\omega$-categorical, stable, but not superstable. (-Counterexample to Lachlan’s Conjecture).

2. (E. Hrushovski, 1997) Take $\alpha$ rational and $F$ growing sufficiently slowly. The generic $M_F$ is $\omega$-categorical, supersimple of finite $SU$-rank and not one-based.

3. (M. E. Pantano, 1995) Take $\alpha$ rational. By letting $F$ grow slowly, we can obtain algebraic closure growing as fast as we like in $M_F$.

4. Can work with relations of higher arity to obtain multiply transitive structures in all of the above.

5. By suitable choice of $F(x)$ for small $x$ we can ensure that, for example, the smallest cycle in $M_F$ is a 5-cycle. This is the only known way of constructing an $\omega$-categorical connected graph whose smallest cycle is a 5-cycle and whose automorphism group is transitive on pairs of adjacent vertices.

6. If $(\mathcal{K}_F, \leq)$ is a free amalgamation class, then $M_F$ does not have the strict order property (- it is $NSOP_4$).

Open Problem: Can algebraic closure grow arbitrarily quickly in stable $\omega$-categorical structures? (In a finite language?)

Strange Problem: Is there a suitable choice of $F$ for all $\alpha$ (- so irrational $\alpha$ not of infinite index)?
Model-theoretic properties: the unconstrained case

- $\delta(B) = |B| - \alpha e(B)$
- $A \leq B$ iff $\delta(A) < \delta(B_1)$ for all $A \subset B_1 \subseteq B$
- $\mathcal{K}_0$: $\emptyset \leq A$
- $(\mathcal{K}_0, \leq)$-generic: $M_0$
- $A \preceq^* B$ iff $\delta(A) \leq \delta(B_1)$ for all $A \subseteq B_1 \subseteq B$
- $\mathcal{K}_0^*$: $\emptyset \preceq^* A$
- $(\mathcal{K}_0^*, \preceq^*)$-generic $M_0^*$

**NOTE:** If $\alpha$ is irrational then $\leq$ and $\preceq^*$ coincide.

1. (J. Baldwin and S. Shelah, 1997; S. Shelah and J. Spencer, 1988) If $0 < \alpha < 1$ is irrational, then $\text{Th}(M_0)$ is stable and has the finite model property. It is the almost-sure theory of finite graphs on $n$ vertices with edge probability $1/n^\alpha$ (as $n \to \infty$).

2. (E. Hrushovski, 1988) If $\alpha$ is rational then $\text{Th}(M_0^*)$ is $\omega$-stable (of infinite Morley rank).

3. (DE, 2003; related earlier work of M. Pourmahdian) Take $\alpha = 1/2$. Then $\text{Th}(M_0)$ is undecidable and has the strict order property.
Sketch of 3.

Work with $\delta(A) = 2|A| - e(A)$.

**IDEA:** Already observed that closure of a pair of points can be arbitrarily large (by taking vertices adjacent to both vertices in the pair). Use this to encode finite graphs $(\Gamma, E)$ into these closures in a uniform way.

This encodes the graph $\Gamma$ (-marked in red) as a graph $A_\Gamma$ (-edges in black). We have $A_\Gamma \in \mathcal{K}_0$ and all vertices of $A_\Gamma$ are in the closure of $a, b$. 
Let $\chi(a, b)$ denote the $L$-formula which says that this picture is accurate (so $V(a, b)$ the set of vertices adjacent to $a, b$ has no edges in it etc.). If $A \in \mathcal{K}_0$ and $A \models \chi(a, b)$, then we interpret a graph in $A$ with vertex set $V(a, b)$ and edges determined by $S(a, b)$.

Given any first-order sentence $\phi$ in the language of graphs we can write down an $L$-formula $\theta(a, b)$ such that for any graph $\Gamma$:

$$\Gamma \models \phi \iff A_\Gamma \models \theta(a, b).$$

**Theorem:** With this notation, there is a finite graph $\Gamma$ which is a model of $\phi$ iff $M_0 \models \exists a, b(\chi(a, b) \land \theta(a, b))$.

**Proof:** ($\Rightarrow$) Use $A_\Gamma \leq M_0$.

($\Leftarrow$) Take such $a, b$. The closure in $M_0$ of $a, b$ is finite, so the graph interpreted in $M_0$ by $V(a, b)$ and $S(a, b)$ is finite. By construction of $\theta$ it is a model of $\phi$. □

This gives undecidability of $Th(M_0)$ by Trakhtenbrot’s Theorem.

For the strict order property note that we can construct a family of finite graphs in which arbitrarily large finite linear orders are uniformly interpretable. Translating this into the $A_\Gamma$, and using compactness, there is a model of $Th(M_0)$ in which an infinite linear order is interpretable (using two parameters).

**Problem:** Does $Th(M_0)$ have the finite model property?
Some references:


