Introduction to Geometric Stability Theory

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0. Prerequisities and Suggested Reading

I will use David Marker's book [**Mar**] as the basic reference for the talks, partly because it's likely to be the easiest one to find. Other presentations are available: the books by Hodges [**Hod**], Poizat [**PoE**] are also excellent (and Poizat in French [**PoF**] is even better). The classic text on model theory is by Chang and Keisler [**ChK**]. To go deeper into geometric stability theory you should look at the books by Baldwin [**Bal**], Buechler [**Bue**] and (more advanced) Pillay [**Pil**]. For basic material on Logic you could look at Cameron's book [**Cam**].

You can also find a variety of lecture notes on Model Theory on the web: the MODNET page [**Mod**] lists a selection of these.

0.1. Logic and Model Theory. Model theory studies and compares mathematical structures from the point of view of what can be said about them in a formal language. Throughout these talks, we will be dealing with first-order languages.

I will assume that you are familiar with the terminology, notation and material in Chapter 1 and Chapter 2.1, 2.2 and 2.3 of [**Mar**]. What follows are some more details about this. I will assume that you understand what is meant by: a first-order language (with equality) L; how to build the formulas of L inductively using connectives and quantifiers starting from atomic formulas; bound and free variables; closed formulas (sentences); L-structures; the notation $\mathcal{M} \models \phi$ 'the formula ϕ is true in the L-structure $\mathcal{M}$' (or '$\mathcal{M}$ is a model of ϕ'); L-theories; the theory $Th(\mathcal{M})$ of an L-structure $\mathcal{M}$; elementary equivalence ($\mathcal{M} \equiv \mathcal{N}$). See [**Mar**], 1.1 and 1.2 for this.

Because formulas are defined inductively, results are often proved by induction on the number of connectives and quantifiers in a formula, with the case of atomic formulas as the base step. A good example of this is the fact that if L-structures $\mathcal{M}$ and $\mathcal{N}$ are isomorphic, then they are elementarily equivalent: see 1.1.10 of [**Mar**].

The starting point for model theory is the Compactness Theorem: if Σ is a set of closed L-formulas such that every finite subset of Σ has a model, then Σ has a model. The two most common proofs of this are using Henkin's method (cf. [**Mar**], 2.1) or ultraproducts

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([**Mar**], 2.5.20, for example). Even if you don't know the proof of this result, I will assume that you have seen some applications of it in examples (such as in 2.1 of [**Mar**]).

DEFINITION 0.1. If $\mathcal{M}$, $\mathcal{N}$ are L-structures and $M \subseteq N$ we say that $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ (or $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$) if whenever $\phi(\bar{x})$ is an L-formula and $\bar{a}$ is a tuple of elements from $\mathcal{M}$, then

$$\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{a}).$$

In this case we write $\mathcal{M} \preceq \mathcal{N}$. More generally an embedding between L-structures is an *elementary embedding* if its image is an elemenary substructure of the range.

Note that we can take ϕ to be a closed formula here, so $\mathcal{M} \preceq \mathcal{N}$ implies $\mathcal{M} \equiv \mathcal{N}$ (exercise: give an example where the converse fails).

It is helpful to think of this in terms of definable sets. Suppose $\mathcal{M}$ is an L-structure, $\bar{b}$ a tuple of parameters from M and $\psi(\bar{x}, \bar{b})$ a formula with parameters $\bar{b}$ and n variables $\bar{x}$. The set $\psi[\mathcal{M}, \bar{b}] = \{\bar{a} \in M^n : \mathcal{M} \models \psi(\bar{a}, \bar{b})\}$ is a *(parameter)-definable subset* of M^n. By definition, if $\mathcal{M} \preceq \mathcal{N}$ then $\psi[\mathcal{N}, \bar{b}] \cap M^n = \psi[\mathcal{M}, \bar{b}]$.

Suppose $(I, \leq)$ is an ordered set (such as the natural numbers, or an ordinal), and for each $i \in I$ we have an L-structure $\mathcal{M}_i$. Suppose further that whenever $i \leq j$ then $M_i \preceq M_j$. Then we can consider $\mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i$ as an L-structure in a natural way, and each $\mathcal{M}_i$ is a substructure of $\mathcal{M}$. It can be shown that:

LEMMA 0.2 (Elementary Chains). *With the above notation $\mathcal{M}_i \preceq \bigcup_{j \in I} \mathcal{M}_j$.*

This is useful for constructing models with particular properties (such as saturation) via a transfinite induction. For a proof, see 2.3.11 of [**Mar**].

THEOREM 0.3 (Tarski-Vaught Test). *Suppose $\mathcal{M}$ is a substructure of the L-structure $\mathcal{N}$. Then $M \preceq \mathcal{N}$ if and only if whenever $\psi(x, \bar{y})$ is an L-formula and $\bar{a}$ is a tuple in $\mathcal{M}$, then there is $d \in \mathcal{N}$ such that $\mathcal{N} \models \psi(d, \bar{a})$ iff there is such a d in $\mathcal{M}$.*

PROOF. See 2.3.5 of [**Mar**]. □

THEOREM 0.4 (Löwenheim-Skolem Theorem). *Suppose T is a set of closed L-formulas which has an infinite model $\mathcal{M}$, and $\kappa \geq |L|$. Then T has a model of cardinality κ.*

PROOF. Extend the language by a set of new constant symbols of size κ. Use the compactness theorem to show that there is an elementary extension of $\mathcal{M}$ of cardinality $\geq \kappa$. Then use the Tarski-Vaught Test to construct an elemenary submodel of this of cardinality κ. Details are in 2.3.7 of [**Mar**]. □

An L-*theory* T is a set of closed L-formulas. If ϕ is any (closed) L-formula with the property that any model of T is a model of ϕ then we say that ϕ is *consequence* of T and write $T \models \phi$. (Some authors require that a theory be *consistent* - that is, has a model - and be closed under consequences: I'm following Marker's usage.) We say that T is *complete* if for every closed L-formula ϕ either ϕ or $\neg\phi$ is a consequence of T. For example, if $T = Th(\mathcal{M})$ for some L-structure $\mathcal{M}$, then T is a complete (consistent!) L-theory: this isn't entirely trivial to prove.

Definition 0.5 (Categoricity). Suppose T is an L-theory with infinite models and κ is an infinite cardinal. We say that T is *κ-categorical* if T has a model of cardinality κ and all such models are isomorphic.

Note that if κ is at least the cardinality of L, then the Löwenheim-Skolem Theorem says that there is some model of cardinality κ.

One of the things which model theory tries to do is this. Given an L-structure $\mathcal{M}$, find a 'nice' subset $T_0 \subseteq Th(\mathcal{M})$ with the property that any formula in $Th(\mathcal{M})$ is a consequence of T_0: we say that T_0 *axiomatizes* $Th(\mathcal{M})$. To show that T_0 has the required property, it is enough to show that it is complete (- exercise in the definitions). If applicable, the following is a very convenient way of showing a theory is complete (see 2.2.6 of [**Mar**]):

Theorem 0.6 (Łos-Vaught Test). *Suppose T is a consistent L-theory with no finite models, and T is κ-categorical for some $\kappa \geq |L|$. Then T is complete.*

Proof. Suppose not. Then there is some closed formula ϕ such that neither ϕ nor $\neg\phi$ is a consequence of T. Thus there are models of $T \cup \{\phi\}$ and $T \cup \{\neg\phi\}$. By the Löwenheim-Skolem Theorem, we can take these to be of cardinality κ. They are both models of T, so are isomorphic; however one is a model of ϕ and the other a model of $\neg\phi$, and this is a contradiction. □

For examples of the use of this, see [**Mar**] 2.2.4 and 2.2.5 (algebraically closed fields of charcteristic p). Also have a look at 2.2.11 of [**Mar**] for a purely algebraic application of this (due to Ax).

The definition of one formula being a consequence of a set of formulas is a *semantic* one: it is phrased in terms of models. It is also possible to give a *syntactic* definition of one L-formula ϕ being *deducible* from a set of L-formulas Σ: written $\Sigma \vdash \phi$. For our purposes, it is not necessary to give the precise definition. A deduction of ϕ from Σ is a finite list of formulas ending in ϕ and obeying certain 'logical rules'. It is checkable line-by-line whether a finite list is a deduction (– at least, under the reasonable assumption that the language is recursive). When this is properly set up the main theorem is:

Theorem 0.7 (Gödel's Completeness Theorem). *For a set of closed L-formulas Σ and a closed L-formula ϕ we have:*

$$\Sigma \vdash \phi \Leftrightarrow \Sigma \models \phi.$$

(Exercise: deduce the Compactness Theorem from this.)

Using this we can give a better explanation of what is meant by a 'nice' axiomatization in the above. We say that a theory T is *decidable* if there is an algorithm which, given a closed formula ϕ, will decide (in a finite amount of time) whether $T \models \phi$ or not. Again, it's not really necessary to have a precise definition of 'algorithm' here: just think of it as something which could be implemented on a digital computer in a standard programming language.

Theorem 0.8. *Suppose L is a recursive language and T_0 is a recursively enumerable set of closed L-formulas which is complete (and consistent). Then T_0 is decidable.*

Here, recursiveness of L just means that we can (in principle) program a computer to recognise when something is a formula; recursive enumerability of T_0 means that the computer can systematically generate all the formulas in T_0: the latter is *a priori* weaker than being able to decide whether any given formula is in T_0.

Proof. The algorithm is this. By the assumed recursiveness, we can program the computer to systematically produce all deductions from T_0. Of course, the computer will have to run for ever to produce all deductions, but any particular deduction will appear after a finite amount of time (but we don't necessarily know how long we will have to wait). Now suppose we are given ϕ. Becuase T_0 is complete, after some finite amount of time we will see a deduction of ϕ or a deduction of $\neg\phi$: as soon as we see one of these, the computer can stop. (Also see [**Mar**], 2.2.8.)

□

For example, the theory of algebraically closed fields of characteristic 0 is a complete recursively enumerable theory in the (recursive) language of rings, so is decidable. Of course, the given algorithm is not particularly practical.

0.2. Set theory. I will assume basic knowledge of 'naïve' Set Theory: Axiom of Choice, Cardinality, Ordinals, Transfinite induction, Cardinals. What's needed can be found in [**Mar**], Appendix A, or [**Cam**].

We will work throughout in ZFC: Zermelo-Fraenkel set theory with the Axiom of Choice.

0.3. Algebra. It will be helpful if you know some basics about (algebraically closed) fields: algebraic independence, transcendence degree.

0.4. Lecture contents.

- Types, quantifier elimination, examples using back-and-forth.
- Closure operations, pregeometries. Algebraic closure and strongly minimal sets/structures. Isomorphism type determined by dimension.
- Morley's Categoricity Theorem: some idea of the Baldwin-Lachlan proof (if time).
- Zilber's trichotomy conjecture. Hrushovski's construction from [**Hru**]: the uncollapsed case and the strongly minimal structures.

1. Types, saturation and quantifier elimination

Sample Motivation: We want to analyse whether certain specific theories are complete and what the definable sets are in certain familiar mathematical structures (we will take algebraically closed fields as a worthy example here). This is a hard problem, and we need to develop some machinery before we can tackle it.

1.1. Types. [This is slightly different to Marker's presentation in 4.1 of [**Mar**]: it will of course be equivalent in the end.]

Suppose L is a first-order language and $\mathcal{M}$ an L-structure. Let $\bar{a} = (a_1, \ldots, a_n) \in M^n$. The *complete type* of $\bar{a}$ in M (in the variables $\bar{x} = (x_1, \ldots, x_n)$) is

$$\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \{\phi(\bar{x}) : \mathcal{M} \models \phi(\bar{a})\}.$$

Notice that $Th(\mathcal{M}) \subseteq \mathrm{tp}^{\mathcal{M}}(\bar{a})$ and that if $\mathcal{M} \preceq \mathcal{N}$ then $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{N}}(\bar{a})$. We are interested in the possible sets of formulas $\mathrm{tp}^{\mathcal{N}}(\bar{b})$ where $\bar{b} \in \mathcal{N} \succeq \mathcal{M}$. We now give a more intrinsic way of looking at this.

If $\mathcal{M}$ is an L-structure and $A \subseteq M$ we define the language $L(A)$ to be L together with new constant symbols $(c_a : a \in A)$. We make $\mathcal{M}$ into an $L(A)$-structure $(\mathcal{M}; A)$ by interpreting the new constant symbol c_a by the element $a \in M$. (We will be less than rigorous in the use of this notation as time progresses; in particular we will not normally distinguish betwee a and the symbol c_a.) We refer to this as 'adding parameters for A' to the language. Notice that if $A = \{a_1, \ldots, a_n\}$ then $Th(\mathcal{M}; A)$ and $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ contain the same information.

Define the *elementary diagram* $\Delta(\mathcal{M})$ of the L-structure $\mathcal{M}$ to be $Th(\mathcal{M}; M)$ (in the language $L(M)$). It is easy to show that if we take a model $\mathcal{N}$ of this, look at the set $\mathcal{M}'$ of interpretations of the new constant symbols $(c_a : a \in M)$ and just consider these in the original language, then $\mathcal{M}'$ is isomorphic to $\mathcal{M}$ and $\mathcal{N}$ is an elementary extension of $\mathcal{M}'$.

LEMMA 1.1. *Suppose $\mathcal{M}$ is an L-structure. A set $p(\bar{x})$ of L-formulas (in variables $\bar{x}$) is of the form $\mathrm{tp}^{\mathcal{N}}(\bar{b})$ for some $\bar{b} \in \mathcal{N} \succeq \mathcal{M}$ iff $p(\bar{x}) \supseteq Th(\mathcal{M})$ and $p(\bar{x})$ is a maximal consistent set of formulas (in the variables $\bar{x}$).*

PROOF. ($\Rightarrow$:) Exercise.

($\Leftarrow$:) We first show that $p(\bar{x}) \cup \Delta(\mathcal{M})$ is consistent. By the Compactness Theorem, it is enough to show that any finite subset of this is consistent. So it suffices to take $\theta(\bar{x}) \in p(\bar{x})$ and $\chi(\bar{m}) \in \Delta(\mathcal{M})$ (where $\bar{m}$ is a tuple of the new constant symbols which we are substituting into the L-formula $\chi(\bar{y})$), and show that $(\exists \bar{x})(\theta(\bar{x}) \wedge \chi(\bar{m}))$ has a model. But because $Th(\mathcal{M}) \subseteq p(\bar{x})$ is consistent, we have $(\exists \bar{x})(\theta(\bar{x})) \in Th(\mathcal{M})$, so in fact $(\mathcal{M}; M) \models (\exists \bar{x})(\theta(\bar{x}) \wedge \chi(\bar{m}))$.

By the consistency and the style of argument preceding the lemma, we get $\bar{b} \in \mathcal{N} \succeq \mathcal{M}$ with $p(\bar{x}) \subseteq \mathrm{tp}^{N}(\bar{b})$. Maximality then gives the required equality. $\square$

So now we can see that the set of types we are interested in is attached to $T = Th(\mathcal{M})$ rather than $\mathcal{M}$. We denote it by $S_n(T)$ (or $S_n^{\bar{x}}(T)$ if we need to indicate the variables being used). By the lemma

$$S_n(T) = \{\mathrm{tp}^{\mathcal{N}}(\bar{a}) : \bar{a} \in \mathcal{N} \models T\}.$$

This is called the (n-th) *Stone space* of T. We say 'space' because of the following remark (which we will not use, but which can be helpful in motivating the terminology and some of the things we do later on).

Remarks 1.2. The set $S_n(T)$ can be made into a topological space by taking as basic open sets the subsets of the form $[\phi(\bar{x})] = \{p(\bar{x}) \in S_n(T) : \phi(\bar{x}) \in p(\bar{x})\}$. Note that the complement of this subset is $[\neg\phi(\bar{x})]$ so we have a basis of clopen sets here. The topology is compact: this is essentially the Compactness Theorem. More details can be found on p.119 of [**Mar**]

We will also look at types over parameters. Suppose $\mathcal{M}$ is an L-structure and $A \subseteq M$. If $\bar{b} \in \mathcal{N} \succeq \mathcal{M}$, the (complete) type of $\bar{b}$ over A (in $\mathcal{N}$) is the set of $L(A)$-formulas:

$$\mathrm{tp}^{\mathcal{N}}(\bar{b}/A) = \{\phi(\bar{x}, \bar{a}) : \mathcal{N} \models \phi(\bar{b}, \bar{a})\}.$$

The set of these is $S_n(Th(\mathcal{M}; A))$, which is usually denoted by $S_n(A)$. This is of course ambiguous notation: but when it's used it is understood that there is in the background a complete theory T and A is a subset of a fixed model of T.

1.2. Saturation. Throughout T is a complete L-theory with infinite models and $\mathcal{M} \models T$.

If $p(\bar{x}) \in S_n(T)$ and there is $\bar{a} \in \mathcal{M}$ with $\mathcal{M} \models p(\bar{a})$ then we say that $p(\bar{x})$ is *realised* in $\mathcal{M}$ (of course we can use the same terminology for arbitrary sets of formulas consistent with T: these are called *types* over T, as opposed to the *complete* types). In general, not every model of T realises all the complete types, but models which have this property are very convenient.

Definition 1.3. Suppose T is a complete L-theory and κ is an infinite cardinal. We say that $\mathcal{M} \models T$ is *κ-saturated* if for every subset $A \subseteq M$ with $|A| < \kappa$ every type in $S_1(A)$ is realised in $\mathcal{M}$.

Remarks 1.4. (1) If $\mathcal{M}$ is κ-saturated then $|M| \geq \kappa$: the $L(M)$-type $\{x \neq a : a \in M\}$ is not realised in M!

(2) If $\mathcal{M}$ is κ-saturated then for every natural number n, $\mathcal{M}$ realises every type in $S_n(A)$, for every A of cardinality less than κ (proof by induction on n).

Theorem 1.5. *Suppose $\mathcal{M}$ is an L-structure and κ is an infinite cardinal. Then $\mathcal{M}$ has an elementary extension which is κ-saturated.*

Proof. We sketch this for $\kappa = \omega = \aleph_0$. See 4.3.12 of [**Mar**] for the general case.

The proof uses the result on elementary chains 0.2 and an *amalgamation property* for elementary extensions:

Fact: If $M_0, M_1, M_2 \models T$ and $f_i : M_0 \to M_i$ (for $i = 1, 2$) are elementary embeddings then there is $\mathcal{N} \models T$ and elementary embeddings $g_i : M_i \to \mathcal{N}$ with $g_1 \circ f_1 = g_2 \circ f_2$.

To see this, note that we can assume without loss that the f_i are inclusions and $M_1 \cap M_2 = M_0$. Then we only need to show that $\Delta(\mathcal{M}_1) \cup \Delta(M_2)$ is consistent, and this is an exercise.

Given the fact, if we have $A \subseteq \mathcal{M}_0 \preceq \mathcal{M}_1$ and $p(x) \in S_1(A)$ realised in some $\mathcal{M}_2 \succeq \mathcal{M}_0$, we can find $\mathcal{N} \succeq M_1$ in which $p(x)$ is realised. Using this repeatedly in a transfinite induction and applying 0.2 at the limit stages, we get that: if $\mathcal{M}_0 \models T$ there is $\mathcal{M}_1 \succeq \mathcal{M}_0$ such that if A is a finite subset of M_0 and $p(x) \in S_1(A)$, then $p(x)$ is realised in $\mathcal{M}_1$. Now repeat ω times and apply 0.2 to get what we want. □

Remarks 1.6. This style of recursive construction combining an amalgamation property with a union-of-a-chain argument occurs throughout model theory, which is why the proof was sketched in this way.

An L-structure $\mathcal{M}$ is called *saturated* if it is $|\mathcal{M}|$-saturated. Such structures have very nice properties. For example letting $\kappa = |M|$:

Strong κ-homogeneity: if $\lambda < \kappa$ and $\bar{a}, \bar{b}$ are λ-tuples in $\mathcal{M}$, then $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{M}(\bar{b})$ iff there is an automorphism α of $\mathcal{M}$ with $\alpha(\bar{a}) = \bar{b}$.

κ-universality: Any model of $Th(\mathcal{M})$ of cardinality smaller than κ can be elementarily embedded in $\mathcal{M}$.

Under set-theoretic hypotheses (such as GCH) it can be shown that any L-structure has a saturated elementary extension. However, even in ZFC it can be shown that any L-structure has an elementary extension with the above properties for arbitrarily large κ, but dropping the requirement that κ be the size of the model. These are referred to as 'big models 'or 'monster models' and it is common for model theorists to take all the models they want to work with as living as elementary submodels of such a model. See the remarks on p.248 of [**Mar**] for further details and references.

1.3. Quantifier elimination.

How should we go about trying to prove the completeness of a given theory? How should we go about trying to describe all the definable subsets of a given structure? We describe one approach to these questions which is often successful (in the cases where they have a good answer). This follows closely Chapter 5 of [**PoE**]. There are variations and generalisations of this approach which are also useful.

Suppose L is a first-order language and let $\mathcal{M}$, $\mathcal{N}$ be two L-structures. Suppose $\bar{a}$, $\bar{b}$ are tuples in $\mathcal{M}$, $\mathcal{N}$ respectively (of the same finite length). A *partial isomorphism* from $\mathcal{M}$ to $\mathcal{N}$ which *connects* $\bar{a}$ and $\bar{b}$ is an isomorphism $f : \langle \bar{a} \rangle_{\mathcal{M}} \to \langle \bar{b} \rangle_{\mathcal{N}}$ with $f(\bar{a}) = \bar{b}$. [Here $\langle \bar{a} \rangle_{\mathcal{M}}$ is the substructure generated by $\bar{a}$ in $\mathcal{M}$.]

Exercise: There is such an f iff $\bar{a}$ and $\bar{b}$ have the same quantifier-free type: that is, whenever $\phi(\bar{x})$ is a quantifier-free L-formula, then $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{b})$. [You need to know what a term is; you can then use induction on length of terms.]

We say that a non-empty set Γ of partial isomorphisms is a *back-and-forth system* connecting $\bar{a}$ and $\bar{b}$ (in $\mathcal{M}$ and $\mathcal{N}$) if it satisfies:

(i) there exists $f \in \Gamma$ with $f\bar{a} = \bar{b}$;
(ii) if $f \in \Gamma$ and $a \in M$ there exists $g \in \Gamma$ extending f and with $a \in \mathrm{dom} g$;
(iii) if $f \in \Gamma$ and $b \in N$ there exists $g \in \Gamma$ extending f and with $b \in \mathrm{im} g$.

Theorem 1.7. *Suppose $\mathcal{M}$, $\mathcal{N}$ are L-structures and n a natural number. Let $\bar{a}$, $\bar{b}$ be n-tuples from $\mathcal{M}$, $\mathcal{N}$ respectively. If there is a back-and-forth system connecting $\bar{a}$ and $\bar{b}$, then $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{N}}(\bar{b})$. If $\mathcal{M}$, $\mathcal{N}$ are ω-saturated, then the converse is true.*

REMARKS 1.8. (1) Here we allow that possibility that $n = 0$: thus the conclusion is that $\mathcal{M}$, $\mathcal{N}$ are elementarily equivalent. The general case can be deduced from this by adding parameters for $\bar{a}$, $\bar{b}$.

(2) Here, (ii) is sometimes referred to as 'forth', and (iii) as 'back.' If (ii) and (iii) are satisfied (and we do not mention $\bar{a}$, $\bar{b}$) then we simply refer to Γ as a back-and-forth system.

(3) The theorem is essentially due to R. Fraïssé: see [**PoE**].

(4) The proof of the forward direction proceeds by showing $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(\bar{b})$ by induction on the number of quantifiers in ϕ. For the converse, let Γ be the set of partial isomorphisms $f : \langle \bar{a}' \rangle_{\mathcal{M}} \to \langle \bar{b}' \rangle_{\mathcal{N}}$ where $\bar{a}'$, $\bar{b}'$ are tuples in $\mathcal{M}$, $\mathcal{N}$ extending $\bar{a}, \bar{b}$ respectively and which have the same type. Show that this is a back-and-forth system.

THEOREM 1.9. *Let T be a consistent L-theory (not necessarily complete). Let n be a positive integer, $\bar{x} = (x_1, \ldots, x_n)$ and Σ a non-empty set of formulas in these variables. Then the following are equivalent:*

(a) *For every $\mathcal{M} \models T$ and n-tuples $\bar{a}, \bar{b} \in \mathcal{M}$, if $\{\sigma(\bar{x}) \in \Sigma : \mathcal{M} \models \sigma(\bar{a})\} = \{\sigma(\bar{x}) \in \Sigma : \mathcal{M} \models \sigma(\bar{b})\}$, then $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{M}}(\bar{b})$;*
(b) *For every L-formula $\psi(\bar{x})$ there is a boolean combination $\theta(\bar{x})$ of formulas in Σ such that $T \vdash (\forall \bar{x})(\psi(\bar{x}) \leftrightarrow \theta(\bar{x}))$.*

REMARKS 1.10. (1) This is most commonly applied where Σ is the set of quantifier-free formulas. In this case, if the conclusion of (b) holds for all n, then we say that T has *quantifier elimination*.
(2) The direction (b) implies (a) is trivial.

Putting the last two results together we get the method for proving completeness and quantifier elimination of a theory:

COROLLARY 1.11. *Let T be a consistent L-theory. The following are equivalent:*

(1) *T is complete and has quantifier elimination.*
(2) *Whenever $\mathcal{M}$ and $\mathcal{N}$ are ω-saturated models of T, then the set Γ of isomorphisms from finitely generated substructures of $\mathcal{M}$ to finitely generated substructures of $\mathcal{N}$ is a back-and-forth system.*

PROOF. (1) $\Rightarrow$ (2) : As T is complete and $\mathcal{M}$, $\mathcal{N}$ are ω-saturated, they realise the same types in $S_n(T)$. In particular, for every $\bar{a} \in \mathcal{M}$ there is $\bar{b} \in \mathcal{N}$ and a partial isomorphism which connects them. So Γ is non-empty. Suppose $f \in \Gamma$ has domain generated by the finite tuple $\bar{a}$ and $f\bar{a} = \bar{b}$. Then by the QE, $\mathrm{tp}^{\mathcal{M}}(\bar{a}) = \mathrm{tp}^{\mathcal{N}}(\bar{b})$. By the converse direction of Theorem 1.7 there is a back-and-forth system connecting $\bar{a}$ and $\bar{b}$. By definition, this is a subset of Γ so it follows easily that Γ satifies (ii) of being a back-and-forth system. The condition (iii) is similar.

(2) $\Rightarrow$ (1) : Suppose $\bar{a}$, $\bar{b}$ are tuples in models $\mathcal{M}_0$, $\mathcal{N}_0$ of T (respectively) which satisfy the same quantifier-free formulas. Let $\mathcal{M}$, $\mathcal{N}$ be ω-saturated elementary extensions of M_0, $\mathcal{N}_0$ respectively. Then there is $f : \langle \bar{a} \rangle_{\mathcal{M}} \to \langle \bar{b} \rangle_{\mathcal{N}}$ in Γ. By Theorem 1.7, it follows that

$\text{tp}^{\mathcal{M}}(\bar{a}) = \text{tp}^{\mathcal{N}}(\bar{b})$ and so $\text{tp}^{\mathcal{M}_0}(\bar{a}) = \text{tp}^{\mathcal{N}_0}(\bar{b})$. Applying this where $\bar{a}$, $\bar{b}$ are empty gives that T is complete. Applying it with $\mathcal{M}_0 = \mathcal{N}_0$ and using Theorem 1.9 gives the QE. □

1.4. Algebraically closed fields. Let L be the language of rings $(+, -, ., 0, 1, =)$. The L-theory ACF (algebraically closed fields) has axioms:

- the field axioms written in this language
- for each $n \geq 1$ the axiom

$$(\forall x_0 \dots x_{n-1})(\exists y)(y^n + x_{n-1}y^{n-1} + \dots + x_1 y + x_0 = 0).$$

If p is a prime, let χ_p be the obvious closed formula which forces the characteristic to be p and let ACF_p be $ACF \cup \{\chi_p\}$. Let ACF_0 be $ACF \cup \{\neg\chi_p : p \text{ prime }\}$.

Theorem 1.12. *(Tarski-Chevalley) If p is a prime or 0 , then ACF_p is complete and has QE.*

Proof. We use Corollary 1.11. So let $\mathcal{M}$, $\mathcal{N}$ be ω-saturated models of ACF_p. A finitely generated substructure here consists of the subring generated by some finite set. Note that any isomorphism between two subrings (of $\mathcal{M}$, $\mathcal{N}$) automatically extends in a unique way to an isomorphism between the subfields which they generate. Thus it will be enough to show that the set Γ of isomorphisms between finitely generated subfields of $\mathcal{M}$ and $\mathcal{N}$ is a back-and-forth system.

First, note that the prime subfields of $\mathcal{M}$ and $\mathcal{N}$ are isomorphic (because they are of the same characteristic). So Γ is non-empty. We show 'forth'. So suppose A, B are finitely generated subfields of $\mathcal{M}$, $\mathcal{N}$ respectively (generated by $\bar{a}$, $\bar{b}$) and $f : A \to B$ is an isomorphism. Let $c \in \mathcal{M}$. We want to find $d \in \mathcal{N}$ and an isomorphism $g : A(c) \to B(d)$ which extends f.

Case 1: c is algebraic over A. Let $F(t) \in A[t]$ (polynomials in t) be the minimum polynomial of c over A. This is irreducible over A, and $A(c)$ is isomorphic to $A[t]$ modulo the ideal generated by F. Clearly $f(F)$ (applying f to the coefficients of F) is irreducible over B. Let d be a root of this in $\mathcal{N}$. Then $B(d)$ is isomorphic to $B[t]$ modulo the ideal generated by $f(F)$. It follows that there is an isomorphism $g : A(c) \to B(d)$ extending f with $g(c) = d$.

Case 2: c is not algebraic over B. In this case, $A(c)$ is the field of rational functions in a single variable over A. So it will suffice to prove that there exists $d \in \mathcal{N}$ which is not algebraic over B: for then there is a unique field isomorphism $A(c) \to B(d)$ which extends f and takes c to d. The existence of d follows from ω-saturation of $\mathcal{N}$. Indeed, consider the type over $\bar{b}$:

$$p(y) = \{G(y) \neq 0 : G(t) \in B[t] \text{ non-zero}\}.$$

[Exercise: why can we think of this as a type over $\bar{b}$ rather than a type over B?]

Any finite subset of this is consistent: for any G there are only finitely many soutions to $G(y) = 0$ and $\mathcal{N}$ is infinite. So by ω-saturation, there is $d \in \mathcal{N}$ with $\mathcal{N} \models p(d)$. Then d is not algebraic over B. □

2. Categoricity and strongly minimal sets

2.1. Overview. Recall that an L-theory T (with infinite models) is κ-categorical if, up to isomorphism, it has a unique model of cardinality κ. We know by the Löwenheim-Skolem Theorem, that a first-order theory cannot pin down the isomorphism type of an infinite structure: we also have to know the cardinality of the structure. Thus, κ-categoricity for some κ is the best that we can hope for. For ω-categorical theories, the main structural result is (cf. 4.4.1 of [**Mar**], for example):

THEOREM 2.1 (Ryll-Nardzewski, Engeler, Svenonius). *Let L be a countable first-order language and T a complete L-theory with infinite models. Then the following are equivalent:*

(1) *T is ω-categorical;*
(2) *For each $n < \omega$, the Stone space $S_n(T)$ is finite;*
(3) *Every countable model of T is ω-saturated;*
(4) *There is a countable model $\mathcal{M}$ of T such that* $\mathrm{Aut}(\mathcal{M})$ *has finitely many orbits on n-tuples, for all $n < \omega$.*

For example, consider the linear ordering $(\mathbb{Q}; \leq)$. The theory T of this includes sentences saying that it is a dense linear order without endpoints. By a theorem of Cantor, any two countable dense linear orders without endpoints are isomorphic, so T is ω-categorical. We can also see this from (4) above. Given $a_1 < \ldots < a_n$ and $b_1 < \ldots < b_n$ in $\mathbb{Q}$, there is a (piecewise-linear) order preserving bijection $\mathbb{Q} \to \mathbb{Q}$ which takes a_i to b_i (for $i \leq n$). Then (4) follows easily from this.

For κ-categorical theories with $\kappa > \omega$ the main result is :

THEOREM 2.2 (Morley's Categoricity Theorem, [**Mor**]). *Suppose L is a countable first-order language and T an L-theory with infinite models. Suppose that T is κ-categorical for some uncountable κ. Then T is λ-categorical for all uncountable λ.*

The proof of this involves producing a structure theory for T. This is the start of geometric stability theory and we will say more about this later on. As part of the proof one shows that a model of T contains a *strongly minimal set* which controls the isomorphism type of the model.

2.2. Algebraic closure. Suppose $\mathcal{M}$ is an L-structure and $A \subseteq M$. We say that a subset $D \subseteq M$ is *A-definable* if there is a formula $\phi(x, \bar{a})$ with parameters from A (that is, an $L(A)$-formula) such that $D = \phi(\mathcal{M}, \bar{a}) = \{b \in M : \mathcal{M} \models \phi(b, \bar{a})\}$. The *algebraic closure* of A (in $\mathcal{M}$) is the union of the finite A-definable subsets of M, denoted by $\mathrm{acl}(A)$.

It is worth noting that if $A \subseteq \mathcal{M} \preceq \mathcal{N}$ then $\mathrm{acl}(A)$ is the same whether it is evaluated in $\mathcal{M}$ or $\mathcal{N}$. If $\phi(x, \bar{a})$ has exactly n solutions in $\mathcal{M}$ then there is an $L(A)$-formula $(\exists^{=n} x)\phi(x, \bar{a})$ which expresses this; so the same is true in $\mathcal{N}$. The formula $\phi(x, \bar{a})$ is *algebraic* if its solution set $\phi(\mathcal{M}, \bar{a})$ is finite.

LEMMA 2.3. *Algebraic closure in $\mathcal{M}$ has the following properties:*

(1) *if $A \subseteq M$ then $A \subseteq \mathrm{acl}(A) = \mathrm{acl}(\mathrm{acl}(A))$;*

(2) *if* $A \subseteq B \subseteq M$ *then* $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$*;*
(3) *if* $b \in \mathrm{acl}(A)$ *then there is a finite subset* A_0 *of* A *with* $b \in \mathrm{acl}(A_0)$.

PROOF. All clear from the definition, except $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$. Suppose c is algebraic over $\mathrm{acl}(A)$. Thus there is a tuple $\bar{b} = (b_1, \ldots, b_n)$ in $\mathrm{acl}(A)$ and a formula $\psi(x, \bar{b})$ with only finitely many solutions (say k) in M, one of which is c. Each $b_i \in \bar{b}$ is algebraic over A, so (by using a $\vee$ of algebraic formulas) there is a $L(A)$-formula $\phi(y, \bar{a})$ with finitely many solutions, amongst which are the b_i. Then the formula

$$(\exists y_1 \ldots y_n)(\psi(x, y_1, \ldots, y_n) \wedge ((\exists^{=k} x)\psi(x, y_1, \ldots, y_n)) \wedge \bigwedge_i \phi(y_i, \bar{a}))$$

has only finitely many solutions, amongst which is c. □

REMARKS 2.4. Properties (1), (2) here say that acl is a *closure operation*; the property (3) says that it is *finitary*.

EXAMPLE 2.5. Let $T = ACF_p$ and $\mathcal{M} \models T$. By QE, a formula $\phi(x, \bar{a})$ is equivalent (modulo T) to a boolean combination of formulas of the form '$F(x) = 0$', where F is a polynomial with coefficients in the subfield generated by $\bar{a}$ (Exercise!). It follows that in this example, algebraic closure in the model-theoretic sense is the same as algebraic closure in the field-theoretic sense.

LEMMA 2.6. *Suppose* $\mathcal{M}$*,* $\mathcal{N}$ *are models of a complete theory* T *and* $f : A \to B$ *is an elementary map between subsets of* $\mathcal{M}$ *and* $\mathcal{N}$ *(meaning: if* $\bar{a}$ *is a tuple in* A*, and* $\phi(\bar{x})$ *is an* L*-formula, then* $\mathcal{M} \models \phi(\bar{a}) \Leftrightarrow \mathcal{N} \models \phi(f\bar{a})$*). Then* f *extends to an elementary map between* $\mathrm{acl}(A)$ *and* $\mathrm{acl}(B)$.

PROOF. One way is to embed $\mathcal{M}$ and $\mathcal{N}$ as elementary submodels of a highly homogeneous model (which we can take as $\mathcal{M}$). Then there is an automorphism γ of $\mathcal{M}$ with $\gamma|A = f$. One then argues that $\gamma\mathrm{acl}(A) = \mathrm{acl}(\gamma(A)) = \mathrm{acl}(B)$.

More directly, suppose $c \in \mathrm{acl}(A)$. Let $\phi(x, \bar{a})$ be an algebraic formula satisfied by c with parameters in A with minimal size of solution set. Minimality means that this *isolates* $\mathrm{tp}(c/A)$: a formula with parameters in A is in $\mathrm{tp}(c/A)$ iff it is implied by $\phi(x, \bar{a})$.

As f is elementary, $\mathcal{N} \models \exists x \phi(x, f\bar{a})$: take $d \in \mathcal{N}$ satisfying it. Again by elementarity, $\phi(x, f\bar{a})$ is algebraic and isolates $\mathrm{tp}(d/B)$. It follows that $f \cup \{(c, d)\}$ is elementary.

A Zorn's lemma argument now shows that f extends to an elementary map $\mathrm{acl}(A) \to \mathrm{acl}(B)$. □

EXERCISE: Use the Ryll-Nardzewski Theorem to show that if $\mathcal{M}$ is ω-categorical, then algebraic closure is *locally finite*: if $A \subseteq \mathcal{M}$ is finite, then $\mathrm{acl}(A)$ is finite (do this for the countable model, then explain why it works in any model).

2.3. Strongly minimal sets. Suppose $\mathcal{M}$ is an L-structure and $D = \phi(\mathcal{M}, \bar{a}) \subseteq M$ is an infinite definable subset of $\mathcal{M}$. We say that D (or $\phi(x, \bar{a})$) is *strongly minimal* if for every $\mathcal{N} \succeq \mathcal{M}$ and every definable subset Y of $\mathcal{N}$, either $\phi(\mathcal{N}, \bar{a}) \cap Y$ or $\phi(\mathcal{N}, \bar{a}) \setminus Y$ is finite. It is an exercise to show that this is equivalent to the condition: whenever $\psi(x, \bar{y})$ is an

L-formula, there is $n < \omega$ (depending on ψ), such that for all $\bar{b} \in \mathcal{M}$, either $\phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ or $\phi(x, \bar{a}) \wedge (\neg\psi(x, \bar{b}))$ has at most n solutions.

If $\mathcal{M}$ itself is strongly minimal we refer to it as a strongly minimal structure: this is a property of $Th(\mathcal{M})$. To connect with Marcus' talks: note that $\mathcal{M}$ is strongly minimal iff the parameter definable subsets of M are the ones which are definable just using formulas involving $=$.

Example 2.7. The following are strongly minimal:

(1) ACF_p;
(2) An infinite set in the language with just $=$;
(3) An infinite vector space over a field (or division ring) F in the language $(+, -, 0, (f_\alpha : \alpha \in F))$ (where f_α is the function: scalar multiplication by α).

For (1), we already noted that any formula $\psi(x, \bar{b})$ is equivalent (modulo T) to a boolean combination of formulas of the form '$F(x) = 0$', where F is a polynomial. Each of these has only finitely many solutions (in any model), so a boolean combination of them has finitely many, or cofinitely many solutions.

We leave (2), (3) as exercises. First axiomatise the theory and prove QE in the given language using Corollary 1.11; consideration of the quantifier-free formulas then gives strong minimality. Note that an ω-saturated model for (3) is an infinite dimensional F-vector space.

In (2) we have $\mathrm{acl}(A) = A$ for all A. In (3), algebraic closure is the same as linear closure over F.

In each of the above examples we have a notion of dimension: in (1) it is transcendence degree, in (2) it is cardinality and in (3) it is vector space dimension. In fact, we have a well-behaved notion of dimension for any strongly minimal set.

Theorem 2.8. *Suppose $\mathcal{M}$ is strongly minimal. Then algebraic closure satisfies the Exchange Property: for $A \subseteq M$ and $a, b \in M$,*

$$\text{if } a \in \mathrm{acl}(A \cup \{b\}) \setminus \mathrm{acl}(A) \text{ then } b \in \mathrm{acl}(A \cup \{a\}).$$

Proof. Suppose $\phi(x, \bar{a}, b)$ is an algebraic formula satisfied by a, where $\bar{a} \in A$. Suppose this has n solutions in $\mathcal{M}$ and consider the $L(A)$-formula $\chi(y)$ given by $(\exists^{=n} x)\phi(x, \bar{a}, y)$. This is satisfied by b and so has infinitely many solutions (otherwise $b \in \mathrm{acl}(A)$ and so $a \in \mathrm{acl}(A)$). So all but finitely many elements of M satisfy it.

If $\chi(y) \wedge \phi(a, \bar{a}, y)$ has only finitely many solutions, then we have $b \in \mathrm{acl}(A \cup \{a\})$ as required.

So suppose this is not the case, and all but k elements of M satisfy it. We obtain a contradiction. By assumption, a satisfies:

$$(\exists y_1 \dots y_k)(\forall y)(\bigwedge_{i=1}^{k}(y \neq y_i) \to ((\exists^{=n} x)\phi(x, \bar{a}, y)) \wedge \phi(x, \bar{a}, y)).$$

But as M is infinite, at most n elements x satisfy this. So $a \in \mathrm{acl}(A)$: contradiction. $\square$

A set X with a finitary closure operation cl on it which satisfies the exchange property is called a *pregeometry* (or finitary matroid). A subset $Z \subseteq X$ is *independent* if for all $z \in Z$ we have $z \notin \mathrm{cl}(Z \setminus \{z\})$. A *basis* of $Y \subseteq X$ is a maximal independent subset of Y. Using Zorn's lemma, such a subset exists, and it follows from the exchange principle that any two bases of Y have the same cardinality: this is called the *dimension* of Y.

Given this, it follows that a strongly minimal structure has associated to it notions of independence and dimension, coming from the operation of algebraic closure. In the case of a vector space, these are the usual linear independence and dimension; in the case of an algebraically closed field, they are algebraic independence and transcendence degree. [You should be able to adapt the proof that any two bases of a vector space have the same cardinality to this more general context.]

Lemma 2.9. *Suppose T is a strongly minimal theory and $A \subseteq \mathcal{M} \models T$. Then there is a unique non-algebraic complete 1-type in $S_1(A)$.*

Proof. Let $p(x)$ consist of non-algebraic formulas $\phi(x, \bar{a})$ with parameters in A. By strong minimality, this is consistent and complete. Any other $q(x) \in S_1(A)$ contains an algebraic formula. □

Corollary 2.10. *Suppose T is strongly minimal $\mathcal{M}, \mathcal{N} \models T$ contain A and $\bar{b} = (b_1, \ldots, b_n)$, $\bar{c} = (c_1, \ldots, c_n)$ are n-tuples in $\mathcal{M}, \mathcal{N}$ respectively which are independent over A (meaning: independent with respect to algebraic closure over A). Then $\mathrm{tp}^{\mathcal{M}}(\bar{b}/A) = \mathrm{tp}^{\mathcal{N}}(\bar{c}/A)$.*

Proof. We can work in a large highly homogeneous model of T which contains $\mathcal{M}$ and $\mathcal{N}$ as elementary submodels. We might as well assume that this is $\mathcal{M}$. We now argue by induction on n. By inductive assumption $\mathrm{tp}(b_1 \ldots b_{n-1}/A) = \mathrm{tp}(c_1 \ldots c_{n-1}/A)$ so (by the strong homogeneity) there is an automorphism γ of $\mathcal{M}$ which fixes all elements of A and sends $(b_1 \ldots b_{n-1})$ to $\bar{c}' = (c_1 \ldots c_{n-1})$. Now γb_n and c_n are non-algebraic over $A\bar{c}'$, so by the lemma, they have the same type over it. Thus $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\gamma\bar{b}/A) = \mathrm{tp}(\bar{c}/A)$ as required. □

Theorem 2.11. *Suppose T is a strongly minimal theory and suppose $\mathcal{M}, \mathcal{N}$ are models of T of the same dimension κ. Then $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.*

Proof. Let $(a_i : i < \kappa)$ and $(\bar{b}_i : i < \kappa)$ be bases of $\mathcal{M}$, $\mathcal{N}$ respectively. By the Corollary, the bijection $a_i \mapsto b_i$ is an elementary map (meaning: $\mathrm{tp}^{\mathcal{M}}((a_i : i < \kappa)) = \mathrm{tp}^{\mathcal{N}}((b_i : i < \kappa))$). But by Lemma 2.6, any elementary map between subsets extends to an isomorphism between their algebraic closures, so in this case, we get an isomorphism between $\mathcal{M}$ and $\mathcal{N}$. □

Corollary 2.12. *Suppose L is a countable language and T is a strongly minimal L-theory. Then T is κ-categorical for all uncountable κ.*

Proof. If $A \subseteq \mathcal{M} \models T$, then $|A| \leq |\mathrm{acl}(A)| \leq |A| + \aleph_0$. So if $\mathcal{M}$ has cardinality $\kappa > \aleph_0$ and $\mathrm{acl}(A) = M$, then $|A| = \kappa$. In particular, a basis of $\mathcal{M}$ has cardinality κ. Now use the above theorem. □

This has all been about strongly minimal structures, but we can adapt most of it to strongly minimal definable sets. Indeed, suppose $\mathcal{M}$ is an L-structure and $D = \phi(\mathcal{M})$ is a strongly minimal set (we will assume that this is without parameters: if necessary, for what follows we can add the necessary parameters for the strongly minimal set to the language).

For $A \subseteq D$ we define $\mathrm{acl}_D(A) = D \cap \mathrm{acl}(A)$. This gives a pregeometry on D, and we can refer to independence, dimension etc. with respect to this.

Analogously with Theorem 2.11 we then have (cf. 6.1.11 of [**Mar**]):

THEOREM 2.13. *Suppose T is a complete L-theory and $\phi(x)$ is a strongly minimal formula (without parameters). If $\mathcal{M}$ and $\mathcal{N}$ are models of T in which the dimensions of $\phi(\mathcal{M})$ and $\phi(\mathcal{N})$ are the same, then there is an elementary bijection $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$.*

3. Morley's Categoricity Theorem and ω-stability

3.1. ω-stability and total transcendence. Suppose T is a complete L-theory and λ is an infinite cardinal. We say that T is *λ-stable* if for every $\mathcal{M} \models T$ and subset $A \subseteq \mathcal{M}$ of cardinality at most λ we have $|S_1(A)| \leq \lambda$.

EXAMPLE 3.1. Suppose T is a strongly minimal L-theory and L is countable. Then T is ω-stable. Indeed, if A is a countable subset of a model of T then $\mathrm{acl}(A)$ is countable. There is a unique non-algebraic complete 1-type over this (by Lemma 2.9); the remaining 1-types over this are determined by formulas of the form '$x = a$' (for $a \in \mathrm{acl}(A)$). Thus

$$|S_1(A)| \leq |S_1(\mathrm{acl}(A))| = |\mathrm{acl}(A)| + 1 \leq \omega.$$

It is easy to show that if T is λ-stable then for each subset A of size at most λ in a model of T, and each natural number n, we have $|S_n(A)| \leq \lambda$.

You might like to think about why, using the definition, $Th(\mathbb{Q}; \leq)$ is not λ-stable for any λ. At least, show that it is not ω-stable.

We say that a complete L-theory T is *totally transcendental* (or just 't.t.') if there *do not exist* a model $\mathcal{M} \models T$, formulas $(\phi_\eta(x, \bar{a}_\eta) : \eta \in 2^{<\omega})$ with parameters in M such that

- for each $\zeta \in 2^\omega$, the set $\{\phi_\eta(x, \bar{a}_\eta) : \eta = \zeta|n\ , n < \omega\}$ is consistent;
- for each $\eta \in 2^{<\omega}$, the formulas $\phi_{\eta 0}(x, \bar{a}_{\eta 0})$ and $\phi_{\eta 1}(x, \bar{a}_{\eta 1})$ are inconsistent.

Explanation: Here $2 = \{0, 1\}$ and $2^{<\omega}$ is the set of finite sequences of 0's and 1's which we think of as the nodes of a binary tree. At each node η there is a a formula $\phi_\eta(x, \bar{a}_\eta)$. The branches of the tree are the infinite sequences $\zeta \in 2^\omega$. The first condition says that as we go along a branch we pick up a consistent set of formulas; the second condition says that the formulas at the successors of any node are inconsistent with each other.

If we have such a 'tree of formulas', there is a countable set A from which all the parameters in the formulas come. On the other hand, going along different branches gives different types: so there are 2^ω types over A. This shows the first part of (cf. 6.2.14 of [**Mar**]):

LEMMA 3.2. *If T is ω-stable then it is totally transcendental. If L is countable, then the converse is also true.*

Note that being t.t. is preserved under adding parameters.

3.2. Prime models. Suppose T is a complete L-theory.

A model $\mathcal{M} \models T$ is a *prime model* of T if for every $\mathcal{N} \models T$ there is an elementary embedding $\mathcal{M} \to \mathcal{N}$. If $A \subseteq \mathcal{M}$, we say that $\mathcal{M}$ is a *prime model over* A if $(\mathcal{M}; A)$ is a prime model of $T(A) = Th(\mathcal{M}; A)$.

Suppose $\phi(\bar{x})$ is an L-formula and $p(\bar{x}) \in S_n(T)$. We say that $\phi(\bar{x})$ *isolates* $p(\bar{x})$ if for every L-formula $\psi(\bar{x})$

$$\psi(\bar{x}) \in p(\bar{x}) \Leftrightarrow T \models (\forall \bar{x})(\phi(\bar{x}) \to \psi(\bar{x})).$$

We say that $p(\bar{x})$ is isolated if some formula isolates it.

The following can be expressed as saying that in a t.t. theory, 'isolated types are dense' (in the Stone space).

LEMMA 3.3. *Suppose T is t.t. and $\theta(\bar{x})$ is a consistent formula. Then there is an isolated type $p(\bar{x})$ with $\theta(\bar{x}) \in p(\bar{x})$.*

PROOF. If $\theta(\bar{x})$ doesn't isolate a complete type, there is an L-formula $\psi(\bar{x})$ such that both $\theta(\bar{x}) \wedge \psi(\bar{x})$ and $\theta(\bar{x}) \wedge (\neg\psi(\bar{x}))$ are consistent. Repeat this argument with both these formulas. Either we can carry on doing this ω times, in which case we have a tree of L-formulas which contradicts t.t.; or we arrive at a formula $\theta(\bar{x}) \wedge \ldots$ which isolates a complete type. □

Exercise: use a similar argument to show that if T is t.t. then there is a model $\mathcal{M} \models T$ and a strongly minimal formula $\phi(x, \bar{a})$ with parameters in $\mathcal{M}$.

THEOREM 3.4. *Suppose T is such that the conclusion of the above lemma holds for $T(A)$ whenever $A \subseteq \mathcal{M} \models T$ (eg. suppose T is t.t.). Then for every $A \subseteq \mathcal{M} \models T$, there is a prime model $\mathcal{M}_A$ of $T(A) = Th(\mathcal{M}; A)$. Moreover, we can choose $\mathcal{M}_A$ so that the type over A of every tuple of elements of $\mathcal{M}_A$ is isolated.*

PROOF. See 4.2.20 of [**Mar**]. □

3.3. Outline of proof of Morley's Theorem. We give a very brief outline of a proof of Morley's Categoricity Theorem. This proof is due to Baldwin and Lachlan [**BaL**] and emphasises the role of strong minimality. Full details can be found in [**Mar**].

For the rest of this section L is a countable language and T is a complete L-theory (with infinite models).

STEP 1: Suppose T is κ-categorical for some uncountable κ. Then T is totally transcendental.

– See 5.2.10 of [**Mar**]. Using an Ehrenfeucht-Mostowski construction, there is a model of cardinality κ in which the number of types over any countable subset which are realized in the model is countable. From this and κ-categoricity, it follows that T is ω-stable.

Terminology: Suppose $\mathcal{M} \prec \mathcal{N} \models T$ and $\mathcal{M} \neq \mathcal{N}$. We say that $(\mathcal{M},\mathcal{N})$ is a *Vaughtian pair* (for T) if there is a non-algebraic formula $\phi(\bar{x})$ with parameters in $\mathcal{M}$ such that $\phi(\mathcal{M}) = \phi(\mathcal{N})$.

STEP 2: If T is κ-categorical for some uncountable κ, then T has no Vaughtian pairs.

– See 5.2.11 of [**Mar**]. This is the hardest part of the proof.

STEP 3: Suppose T is totally transcendental and has no Vaughtian pairs. Then T is λ-categorical for every uncountable λ.

Let $\mathcal{M}_0$ be a prime model of T. We first argue that there is a strongly minimal formula $\phi(x, \bar{a})$ with parameters in $\mathcal{M}_0$. Indeed, using total transcendence, we show (as in the exercise above) that there is a non-algebraic formula $\phi(x, \bar{a})$ with parameters in $\mathcal{M}_0$ such that for any other $L(\mathcal{M}_0)$-formula $\psi(x, \bar{b})$, either $\phi(x, \bar{a}) \wedge \neg\psi(x, \bar{b})$ or $\phi(x, \bar{a}) \wedge \psi(x, \bar{b})$ is algebraic. The fact that T has no Vaughtian pairs then implies that $\phi(x, \bar{a})$ is strongly minimal: see 6.1.14 and 6.1.15 of [**Mar**].

Suppose $\mathcal{M} \models T$ has cardinality $\lambda > \omega$. We can suppose $\mathcal{M}_0 \preceq \mathcal{M}$. Consider $A = \phi(\mathcal{M})$. There is a prime model $\mathcal{M}_1$ over this and we can assume $\mathcal{M}_1 \preceq \mathcal{M}$. Now $\phi(\mathcal{M}_1) = \phi(\mathcal{M})$ so as there are no Vaughtian pairs, $\mathcal{M}_1 = \mathcal{M}$. So $\mathcal{M}$ is prime over $\phi(\mathcal{M})$ and it follows that $|\phi(\mathcal{M})| = \lambda$. So $\phi(\mathcal{M})$ has dimension λ.

If $\mathcal{N} \models T$ also has cardinality λ then again we can assume $\mathcal{M}_0 \preceq \mathcal{N}$ and $\phi(\mathcal{N})$ has dimension λ. So by Theorem 2.13, there is an elementary bijection $f : \phi(\mathcal{M}) \to \phi(\mathcal{N})$ (over the parameters of ϕ). As $\mathcal{M}$ is prime over $\phi(\mathcal{M})$ this extends to an elementary embedding $g : \mathcal{M} \to \mathcal{N}$. But $g\mathcal{M} = \mathcal{N}$: otherwise we have a Vaughtian pair. So $\mathcal{M}$ and $\mathcal{N}$ are isomorphic.

4. Hrushovski constructions

A conjecture of Zilber from around 1980 asserted that the 'classical' examples of strongly minimal structures – pure set; vector spaces; algebraically closed fields – are 'essentially' the only examples of strongly minimal structures.

This conjecture was refuted by Ehud Hrushovski in 1988 (in an unpublished manuscript which was incorporated into [**Hru**]). In the rest of my talks, I will describe Hrushovski's construction of a 'new' strongly minimal set, in much the same way as it appears in [**Hru**]; a general framework for these types of constructions can be found in Wagner's article [**Wag**]. Zilber's lectures will discuss contexts in which his conjecture remains true, and ways in which Hrushovski's example can be seen as part of 'classical' mathematics.

We want to build a strongly minimal set: a structure with a dimension on it. We build it from finite structures each of which carries in a natural way a dimension.

4.1. Predimension and dimension.

The notation will be cumulative for the rest of the talks.

Throughout this section L will be the language having just a 3-ary relation symbol R. We work with L-structures which are models of the following sentences T':

$$(\forall x_1x_2x_3)(R(x_1,x_2,x_3) \rightarrow \bigwedge_{i\neq j}(x_i \neq x_j))$$

and for every permutation π of $1,2,3$:

$$(\forall x_1x_2x_3)(R(x_1,x_2,x_3) \leftrightarrow R(x_{\pi 1},x_{\pi 2},x_{\pi 3})).$$

Thus in a model A of these, we can regard the interpretation R^A of R as a set of 3-subsets of A (rather than a set of triples).

If $B \models T'$ is finite define the *predimension* of B to be

$$\delta(B) = |B| - |R^B|.$$

If $B \subseteq A \models T'$ we can regard B as a substructure of A and if B is finite we can consider $\delta(B)$. We let $\bar{\mathcal{C}}_0$ be the class of structures A with the property that

$$\delta(B) \geq 0 \text{ for all finite } B \subseteq A.$$

Let $\mathcal{C}_0$ be the finite structures in $\bar{\mathcal{C}}_0$.

It is easy to see that there is an L-theory T_0 whose models are precisely the structures in $\bar{\mathcal{C}}_0$.

Lemma 4.1 (Submodularity). *If $A \in \bar{\mathcal{C}}_0$ and B, C are finite subsets of A, then*

$$\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C).$$

There is equality iff $R^{B\cup C} = R^B \cup R^C$ (in which case we say that B, C are freely amalgamated over their intersection).

Proof. Note that the left-hand side minus the right-hand side of the inequality is:

$$(|B\cup C| - (|B| + |C| - |B\cap C|)) - (|R^{B\cup C}| - |R^B| - |R^C| + |R^{B\cap C}|).$$

As $R^{B\cap C} = R^B \cap R^C$, this is equal to

$$-(|R^{B\cup C}| - |R^B \cup R^C|).$$

Hence the result. □

If $A \subseteq B \in \bar{\mathcal{C}}_0$ is finite and for all finite B' with $A \subseteq B' \subseteq B$ we have $\delta(A) \leq \delta(B')$, then we say that A is *self-sufficient* in B and write $A \leq B$.

Lemma 4.2. *Suppose $B \in \mathcal{C}_0$.*

1. *If $A \leq B$ and $X \subseteq B$, then $A \cap X \leq X$.*
2. *If $A \leq B$ and $B \leq C \in \mathcal{C}_0$ then $A \leq C$.*
3. *If $A_1, A_2 \leq B$ then $A_1 \cap A_2 \leq B$.*

Proof. (1) Let $A \cap X \subseteq Y \subseteq X$. Then

$$\delta(A \cup Y) \leq \delta(A) + \delta(Y) - \delta(Y \cap A).$$

So as $A \cap X = A \cap Y$ we have:

$$\delta(Y) - \delta(X \cap A) \geq \delta(A \cup Y) - \delta(A) \geq 0.$$

(2) Let $A \subseteq X \subseteq C$. As $B \leq C$ we have $X \cap B \leq X$ (by (1)) so $\delta(X \cap B) \leq \delta(X)$. Also, $A \subseteq X \cap B \subseteq B$ so $\delta(A) \leq \delta(X \cap B)$, by $A \leq B$. So $\delta(A) \leq \delta(X)$.

(3) By (1) we have $A_1 \cap A_2 \leq A_2$. So $A_1 \cap A_2 \leq B$, using (2). □

If $B \in \bar{\mathcal{C}}_0$ and $A \subseteq B$ then we write $A \leq B$ when $A \cap X \leq X$ for all finite $X \subseteq B$. It can be checked that the above lemma still holds.

If X is a finite subset of $B \in \bar{\mathcal{C}}_0$ then there is a finite set C with $X \subseteq C \subseteq B$ and $\delta(C)$ as small as possible. Then by definition, $C \leq B$. Now take C as small as possible. By (3) of the above lemma, C is uniquely determined by X: it is the intersection of all self-sufficient subsets of B which contain X. We refer to this as the *self-sufficient closure* of X in B and denote it by $\mathrm{cl}_B^{\leq}(X)$. Write $d_B(X) = \delta(\mathrm{cl}_B^{\leq}(X))$. By the above discussion:

$$d_B(X) = \min\{\delta(C) : X \subseteq C \subseteq_f B\}.$$

(Where $C \subseteq_f B$ means C is a finite subset of B.)

This is the *dimension* of X in B. It is clear that if $X \subseteq Y \subseteq_f B$ then $d_B(X) \leq d_B(Y)$.

Exercise: Show that self-sufficient closure is a closure operation, but that it does not necessarily satisfy the exchange property.

LEMMA 4.3. *If X, Y are finite subsets of $B \in \bar{\mathcal{C}}_0$ then*

$$d_B(X \cup Y) \leq d_B(X) + d_B(Y) - d_B(X \cap Y).$$

PROOF. Let X', Y' be the self-sufficient closures of X and Y in B. Then

$$d_B(X \cup Y) = d_B(X' \cup Y') \leq \delta(X' \cup Y') \leq \delta(X') + \delta(Y') - \delta(X' \cap Y').$$

Now, $X \cap Y \subseteq X' \cap Y'$ and the latter is self-sufficient in B. So $d_B(X \cap Y) \leq d_B(X' \cap Y') = \delta(X' \cap Y')$. The result follows. □

REMARKS 4.4. From the proof we can read off when we have equality in the above. Suppose for simplicity that X, Y are self-sufficient. Then there is equality in the lemma iff $X \cup Y \leq B$ and X, Y are freely amalgamated over their intersection.

We now relativise the dimenson function. Suppose $B \in \bar{\mathcal{C}}_0$ and $\bar{a}$ is a tuple of elements in B and C a finite subset of B. Define the dimension of $\bar{a}$ over C to be:

$$d_B(\bar{a}/C) = d_B(\bar{a}C) - d_B(C).$$

(Where $\bar{a}C$ denotes the union of C and the elements in $\bar{a}$.)

LEMMA 4.5. *If $\bar{a}$, $\bar{b}$ are tuples in $B \in \bar{\mathcal{C}}_0$ and C is a finite subset of B then:*

1. $d_B(\bar{a}\bar{b}/C) = d_B(\bar{a}/\bar{b}C) + d_B(\bar{b}/C)$.
2. $d_B(\bar{a}\bar{b}/C) \leq d_B(\bar{a}/C) + d_B(\bar{b}/C)$.
3. *If $C' \subseteq C$ then $d_B(\bar{a}/C') \geq d_B(\bar{a}/C)$.*

PROOF. Drop the subscript B here. (1) is by definition and (2) follows from (1) and (3).
To prove (3), let $A' = \mathrm{cl}^{\leq}(\bar{a}C')$. Then

$$d(\bar{a}C) = d(A' \cup C) \leq d(A') + d(C) - d(A' \cap C) \leq d(A') + d(C) - d(C').$$

Rearranging gives what we want. $\square$

We can extend this to arbitrary $C \subseteq B$. We define $d_B(\bar{a}/C)$ to be the minimum of $d(\bar{a}/C')$ for $C' \subseteq_f C$. By (3), this is harmless if C is actually finite. It can then be shown that the above lemma holds for arbitrary C.

LEMMA 4.6. *Suppose $C \leq B \in \bar{\mathcal{C}}_0$ and $a \in B$. Then $0 \leq d_B(a/C) \leq 1$. Moreover $d_B(a/C) = 1$ iff $\{a\} \cup C \leq B$ and there is no 3-set in R^B which consists of a and two points of C.*

PROOF. We can assume C is finite (Ex: why?) and $a \notin C$. Then $d(aC) \leq \delta(aC) \leq \delta(a) + \delta(C) = 1 + \delta(C) = 1 + d(C)$. The rest follows from looking at where we have equality, as in the above remarks. $\square$

We now define another closure operation, which does satisfy exchange.

Suppose $C \subseteq B \in \bar{\mathcal{C}}_0$. The *d-closure* of C in B is:

$$\mathrm{cl}^d_B(C) = \{a \in B : d_B(a/C) = 0\}.$$

THEOREM 4.7. *If $B \in \bar{\mathcal{C}}_0$ then d-closure in B is a finitary closure operation which satisfies the exchange condition.*

PROOF. Note that if $d_B(a/C) = 0$ then $d_B(a/C') = 0$ for some finite $C' \leq C$. So cl^d is finitary. Clearly $C \subseteq \mathrm{cl}^d_B(C)$ and if $C \subseteq D$ then $\mathrm{cl}^d_B(C) \subseteq \mathrm{cl}^d(D)$. It remains to show $\mathrm{cl}^d(\mathrm{cl}^d(C)) = \mathrm{cl}^d(C)$. We can assume that C is finite. If $a \in \mathrm{cl}^d(\mathrm{cl}^d(C))$ there is a tuple $\bar{b} = (b_1, \ldots, b_n)$ in $\mathrm{cl}^d_B(C)$ with $d_B(a/\bar{b}C) = 0$. But then (using Lemma 4.5)

$$d(a/C) = d(a/\bar{b}C) + d(\bar{b}/C) \leq 0 + \sum_i d(b_i/C) = 0.$$

So $a \in \mathrm{cl}^d_B(C)$.

For exchange, suppose $a \in \mathrm{cl}^d_B(C \cup \{b\}) \setminus \mathrm{cl}^d(C)$. By the above lemma, $d(a/C) = 1$ and $d(a/Cb) = 0$. Then using Lemma 4.5

$$1 = d(ab/C) = d(b/aC) + d(a/C) = d(b/aC) + 1.$$

So $b \in \mathrm{cl}^d_B(aC)$, as required. $\square$

Thus if $B \in \bar{\mathcal{C}}_0$ then we have a pregeometry $(B; \mathrm{cl}^d_B)$.

LEMMA 4.8. *A finite subset $Z = \{z_1, \ldots, z_n\}$ of B is independent in the pregeometry $(B; \mathrm{cl}^d_B)$ iff $d(Z) = |Z|$. Thus, dimension in the sense of the pregeometry is given by d_B.*

Proof. If Z is not independent, then $\mathrm{cl}^d_B(Z) = \mathrm{cl}^d_B(Z_0)$ for some proper subset Z_0 of Z. Then $d(Z) = d(Z_0) \leq |Z_0|$. For the converse, note that

$$d(Z) = d(z_n/z_{n-1}\dots z_1) + d(z_{n-1}/z_{n-2}\dots z_1) + \dots + d(z_1).$$

So if this is $< n$ then $d(z_i/z_{i-1}\dots z_1) = 0$ for some $i \leq n$. Then $z_i \in \mathrm{cl}^d_B(z_1 \dots z_1)$ and Z is not independent. □

4.2. Amalgamation. Suppose B_1, B_2 are structures in $\bar{\mathcal{C}}_0$ (or indeed, just models of T') with a common substructure A. We can assume without loss of generality that $A = B_1 \cap B_2$. We form another structure E with domain $E = B_1 \cup B_2$ and relations $R^E = R^{B_1} \cup R^{B_2}$. We refer to this as the *free amalgam* of B_1 and B_2 over A.

Lemma 4.9 (Free amalgamation lemma). *Suppose $B_1, B_2 \in \bar{\mathcal{C}}_0$ have a common substructure A. Suppose that $A \leq B_1$. Then the free amalgam E of B_1 and B_2 over A is in $\bar{\mathcal{C}}_0$ and $B_2 \leq E$.*

Proof. Note that the condition of being in $\bar{\mathcal{C}}_0$ is equivalent to the empty set being self-sufficient. So it suffices to prove $B_2 \leq E$, for then $\emptyset \leq B_2 \leq E$, and $\emptyset \leq E$ follows.

Let X be a finite subset of E. Write $X_i = X \cap B_i$ and $X_0 = X \cap A$. We want to show that $X_2 \leq X$, so let $X_2 \subseteq Y \subseteq X$. Now, X is the free amalgam of X_1 and X_2 over X_0 so Y is the free amalgam over X_0 of X_2 and $Y \cap X_1$, whence $\delta(Y) = \delta(Y \cap X_1) + \delta(X_2) - \delta(X_0)$. Thus

$$\delta(Y) - \delta(X_2) = \delta(Y \cap X_1) - \delta(X_0).$$

As $A \leq B_1$ we have $X_0 \leq X_1$. So as $X_0 \subseteq Y \cap X_1 \subseteq X_1$, the above is ≥ 0. Thus $\delta(Y) \geq \delta(X_2)$, as required. □

Of course, if $A \leq B_2$ here, then we also obtain $B_1 \leq E$ (by symmetry of the argument). However, we can usefully obtain something sightly stronger.

Suppose m is a natural number. Let $Y \subseteq Z \in \bar{\mathcal{C}}_0$. Write $Y \leq^m Z$ to mean that $\delta(Y) \leq \delta(Z')$ whenever $Y \subseteq Z' \subseteq Z$ and $|Z' \setminus Y| \leq m$. It is easy to check that Lemma 4.2 holds with $\leq$ replaced by $\leq^m$ throughout (the same proof works).

Lemma 4.10 (Strong free amalgamation lemma). *Suppose $B_1, B_2 \in \bar{\mathcal{C}}_0$ have a common substructure A. Suppose that $A \leq^m B_1$ and $A \leq B_2$. Then the free amalgam E of B_1 and B_2 over A is in $\bar{\mathcal{C}}_0$ and $B_2 \leq^m E$ and $B_1 \leq E$.*

Proof. By the previous lemma we have $\emptyset \leq B_1 \leq E$ and so $E \in \bar{\mathcal{C}}_0$. The proof that $B_2 \leq^m E$ just requires careful inspection of the above proof. □

4.3. The uncollapsed generic.

Theorem 4.11 (The generic structure for $(\mathcal{C}_0; \leq)$). *There is a countable $\mathcal{M} \in \bar{\mathcal{C}}_0$ satisfying the following properties:*

(C1): $\mathcal{M}$ is the union of a chain of finite substructures
$B_1 \leq B_2 \leq B_3 \leq \dots$ all of which are in $\mathcal{C}_0$.

(C2): *If $A \leq \mathcal{M}$ is finite and $A \leq B \in \mathcal{C}_0$, then there is an embedding $f : B \to \mathcal{M}$ with $f(B) \leq \mathcal{M}$ and which is the identity on A.*

Moreover $\mathcal{M}$ is uniquely determined up to isomorphism by these two properties and is $\leq$-homogeneous (meaning: any isomorphism between finite self-sufficient substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

Proof. *The construction:* First, note that any countable structure in $\bar{\mathcal{C}}_0$ satisfies (C1). To achieve C2, we construct the B_i inductively so that the following (which is equivalent to C2) holds:

($C2'$) If $A \leq B_i$ and $A \leq B \in \mathcal{C}_0$ then there is $j \geq i$ and a $\leq$-embedding $f : B \to B_j$ which is the identity on A.

Note that there are countably many isomorphism types of $A \leq B$ in $\mathcal{C}_0$. A standard 'organisational' trick allows us to show that we can just do one instance of the problem in ($C2'$). But this is what amalgamation does for us: we have $A \leq B_i$ and $A \leq B$ so let B_{i+1} be the free amalgam of B_i and B over A. Then $B_i \leq B_{i+1}$ and $B \leq B_{i+1}$.

Uniqueness: Suppose $\mathcal{M}$ and $\mathcal{M}'$ satisfy these properties. One shows that the set of isomorphisms $A \to A'$ where $A \leq \mathcal{M}$ and $A' \leq \mathcal{M}'$ are finite is a back-and-forth system. The 'moreover' part follows. □

The structure $\mathcal{M}$ is referred to as the *generic structure* for the *amalgamation class* $(\mathcal{C}_0; \leq)$.

We want to understand $Th(\mathcal{M})$.

4.4. Model theory of $\mathcal{M}$. We want to axiomatize $Th(\mathcal{M})$ and understand types. Recall that T_0 is the set of axioms for the class $\bar{\mathcal{C}}_0$, and $(C1)$ holds in any countable model of these.

The condition in $(C2)$ is not (*a priori*) first-order: how can we express 'for all $A \leq \mathcal{M}$' and '$f(B) \leq \mathcal{M}$? The trick is to replace $\leq$ here by the approximations $\leq^m$.

Note that for each m, and each n-tuple of variables $\bar{x}$ there is a formula $\psi_{m,n}(\bar{x})$ with the property that for every $C \in \bar{\mathcal{C}}_0$ and n-tuple $\bar{a}$ in C we have:

$$\bar{a} \leq^m C \Leftrightarrow C \models \psi_{m,n}(\bar{a}).$$

Suppose $A \leq B \in \mathcal{C}_0$. Let $\bar{x}$, $\bar{y}$ be tuples of variables with $\bar{x}$ corresponding to the distinct elements of A and $\bar{y}$ corresponding to the distinct elements of $B \setminus A$. Let $D_A(\bar{x})$ and $D_{A,B}(\bar{x}, \bar{y})$ denote the basic diagrams of A and B respectively. Suppose A, B are of size n, k respectively. For each m let $\sigma^m_{A,B}$ be the closed L-formula:

$$\forall \bar{x} \exists \bar{y} (D_A(\bar{x}) \wedge \psi_{m,n}(\bar{x}) \to D_{A,B}(\bar{x}, \bar{y}) \wedge \psi_{m,k}(\bar{x}, \bar{y})).$$

Let T consist of T_0 together with these $\sigma^m_{A,B}$.

Theorem 4.12. *We have that $\mathcal{M} \models T$ and T is complete. Moreover, n-tuples $\bar{c}_1$, $\bar{c}_2$ in models $\mathcal{M}_1, \mathcal{M}_2$ of T have the same type iff $\bar{c}_1 \mapsto \bar{c}_2$ extends to an isomorphism between $\mathrm{cl}^{\leq}_{\mathcal{M}_1}(\bar{c}_1)$ and $\mathrm{cl}^{\leq}_{\mathcal{M}_2}(\bar{c}_2)$.*

Proof. *Step 1:* $\mathcal{M} \models T$.
We show $\mathcal{M} \models \sigma^m_{A,B}$. So suppose $A' \leq^m \mathcal{M}$ is isomorphic to A. We have to find $B' \leq^m \mathcal{M}$ isomorphic to B (over A). Let $C = \mathrm{cl}^{\leq}_{\mathcal{M}}(A')$. Let E be the free amalgam of C and B over A (which we identify with A'), and use Lemma 4.10. Then $C \leq E$, so we can use $(C2)$ in $\mathcal{M}$ to get a $\leq$-embedding $f : E \to \mathcal{M}$ which is the identity on C. Then $A' \leq fB \leq^m fE \leq \mathcal{M}$: so $B' = fB$ is what we want.

Step 2: If $\mathcal{N} \models T$ is ω-saturated, then $\mathcal{N}$ satisfies $(C2)$.
Suppose $A \leq \mathcal{N}$ is finite and $A \leq B$. Let $\bar{a}$ enumerate A and $n = |B|$. By the $\sigma^m_{A,B}$, (and compactness) the collection of formulas $\{D_{A,B}(\bar{a}, \bar{y}) \wedge \psi_{m,n}(\bar{a}\bar{y}) : m < \omega\}$ is consistent. So as $\mathcal{N}$ is ω-saturated we get $\bar{b}$ in $\mathcal{N}$ which satisfies all of them. Then $\bar{a}\bar{b} \leq \mathcal{N}$ and this gives what we want.

It then follows easily that if $\mathcal{N}_1$, $\mathcal{N}_2$ are ω-saturated models of T, then the set of isomorphisms between finite $\leq$-substructures of $\mathcal{N}_1$ and $\mathcal{N}_2$ is a back-and-forth system (– we also need to know that any finite subset is contained in a finite $\leq$-subset). Using Theorem 1.7 this gives the 'if' direction in the statement. For the converse, note that if two tuples have the same type, then so do their self-sufficient closures (as these are part of the algebraic closure). □

4.5. ω-stability of $\mathcal{M}$. $\mathcal{M}$ is the generic structure for $(\mathcal{C}_0; \leq)$ as in the previous section and we will let $T = Th(\mathcal{M})$ (this is a harmless change of notation).

Suppose $\mathcal{M}' \models T$ is ω-saturated $B \leq \mathcal{M}'$ and $\bar{a}$ is a tuple in $\mathcal{M}'$. There is a finite $C \leq B$ with $d(\bar{a}/B) = d(\bar{a}/C)$ and we can assume that $\mathrm{cl}(\bar{a}C) \cap B = C$ (– if not, replace C by this intersection).

Claim: $\mathrm{cl}^{\leq}(\bar{a}C) \cup B \leq \mathcal{M}'$ and is the free amalgam of $\mathrm{cl}^{\leq}(\bar{a}C)$ and B over C.

Proof of claim: Let $A = \mathrm{cl}^{\leq}(\bar{a}C)$.

Proof of Claim: Let $A = \mathrm{cl}(\bar{a}C)$. It suffices to prove the claim when B is finite (– by considering finite closed subsets of the original B). By definition of δ if A, B are not freely amalgamated over C then $\delta(\mathrm{cl}^{\leq}(\bar{a}B)) \leq \delta(A \cup B) < \delta(A) + \delta(B) - \delta(C)$, which, after rearranging the inequality, contradicts the choice of C. We have a similar contradiction if $\delta(\mathrm{cl}^{\leq}(\bar{a}B)) < \delta(A \cup B)$, thus $A \cup B \leq \mathcal{M}'$. $\square_{Claim}$

So $\mathrm{tp}(\bar{a}/B)$ is determined by C and the isomorphism type of $\mathrm{cl}(\bar{a}C)$. So the number of 1-types over B is at most $max(\aleph_0, |B|)$. Thus T is λ-stable for all infinite λ.

Remarks 4.13. If $A \leq \mathcal{M}' \models T$ is finite then $\mathrm{acl}(A) = \mathrm{cl}^{\leq}(A)$. Indeed, as $\mathrm{cl}^{\leq}(A)$ is finite, we have $\supseteq$. On the other hand if $b \in \mathcal{M}' \setminus \mathrm{cl}^{\leq}(A)$, let $B' = \mathrm{cl}^{\leq}(bA)$ and $A' = \mathrm{cl}^{\leq}(A)$. We can assume that $\mathcal{M}'$ is ω-saturated, so $(C2)$ holds in $\mathcal{M}'$. By considering the free amalgam of copies of B' over A' we obtain infinitely many elements of $\mathcal{M}'$ with the same type as b over A'.

Note that by Lemma 4.6, if $a, a' \in \mathcal{M}'$ and $d(a/B) = d(a'/B) = 1$ then $\mathrm{tp}(a/B) = \mathrm{tp}(a'/B)$.

Summary: We have built an ω-stable structure $\mathcal{M}$ which has a predimension on it; moreover elements of dimension 1 over a set have the same type over the set. HOWEVER, $\mathcal{M}$ is not strongly minimal: there are types of dimension 0 which are not algebraic. To produce a strongly minimal structure by these methods requires an extra twist.

[An aside:

If you know about forking, here is a characterization of it in models of T.

THEOREM 4.14. *If* $A, B, C \subseteq \mathcal{M}' \models T$ *then* $A \downarrow_C B$ *iff*

- $\mathrm{cl}^{\leq}(AC) \cap \mathrm{cl}^{\leq}(BC) = \mathrm{cl}^{\leq}(C)$
- $\mathrm{cl}^{\leq}(AC)$ *and* $\mathrm{cl}^{\leq}(BC)$ *are freely amalgamated over* $\mathrm{cl}^{\leq}(C)$
- $\mathrm{cl}^{\leq}(ABC) = \mathrm{cl} \leq (AC) \cup \mathrm{cl}^{\leq}(BC)$.

Expressed in a different way, if $\bar{a}$ *is a tuple in* $\mathcal{M}'$, *then* $\bar{a} \downarrow_C B$ *iff* $d(\bar{a}/C) = d(\bar{a}/B)$ *and* $\mathrm{acl}(\bar{a}C) \cap \mathrm{acl}(B) = \mathrm{acl}(C)$.

Sketch of Proof. Assuming the 3 conditions hold. To simplify the notation we can assume that A, B are closed and have intersection C and we can assume that $\mathcal{M}'$ is highly saturated. We show that $\mathrm{tp}(A/B)$ does not divide over C. Suppose $(B_i : i < \omega)$ is a sequence of translates of B over C. Let X be the $\leq$-closure of the union of these and let Y be the free amalgam of X and A over C. As $B_i \leq X$ we have that A and B_i are freely amalgamated over C and $A \cup B_i \leq Y$. We may assume that $Y \leq \mathcal{M}'$. If A' denotes the copy of A in Y then $\mathrm{tp}(A'B_i) = \mathrm{tp}(AB)$ for each i.

For the converse, we can use the fact that algebraic closure in M_1 is self-sufficient closure to obtain the first bullet point if $A \downarrow_C B$. Moreover, we can assume as before that A, B are closed and have intersection C. To simplify the argument, assume also that A, B are finite. Let $(B_i : i < \omega)$ be a sequence of translates of B over A which are freely amalgamated over C and such that the union of any subcollection of them is self-sufficient in M_1. Suppose for a contradiction that A, B are not freely amalgamated over C. Then the same is true of A and B_i and there is $s > 0$ such that $\delta(A \cup B_i) = \delta(A) + \delta(B_i) - \delta(C) - s$ for all i. Then one computes that

$$\delta(A \cup \bigcup_{i=1}^{r} B_i) \leq \delta(\bigcup_{i=1}^{r} B_i) + \delta(A) - C - rs.$$

If r is large enough, this contradicts $\bigcup_{i=1}^{r} B_i \leq M_i$. The third bullet point is similar. □

You can use this to show that the 1-type of dimension 1 is a regular type of U-rank ω. It can also be shown that $Th(\mathcal{M})$ has Morley rank ω.]

4.6. The strongly minimal case.

All of this is taken from [**Hru**].

DEFINITION 4.15. (1) We say that $X \leq Y \in \mathcal{C}_0$ is an *algebraic extension* if $X \neq Y$ and $\delta(X) = \delta(Y)$.

(2) An algebraic extension $X \leq Y$ is *simply algebraic* if there does not exists Y' with $X \subset Y' \subset Y$ and $\delta(X) = \delta(Y')$.

(3) A simply algebraic extension $X \leq Y$ is *minimally simply algebraic* (msa) if there does not exist $X' \subset X$ such that $X' \leq X' \cup (Y \setminus X)$ is simply algebraic.

Exercise: Construct some examples of msa extensions.

REMARKS 4.16. (1) If $X \leq Y$ is an algebraic extension then there exist $X = X_1 \leq X_2 \leq \ldots \leq X_n = Y$ such that each $X_i \leq X_{i+1}$ is simply algebraic.

(2) If $X \leq Y$ is simply algebraic, let $X_0 \subseteq X$ be the elements of X which lie is a relation in $R^Y \setminus R^X$. Then $X_0 \leq X_0 \cup (Y \setminus X)$ is msa.

[Explanation if it makes sense, otherwise ignore: If $X \leq Y \leq \mathcal{M}' \models T$ is simply algebraic then $\mathrm{tp}(Y/X)$ is a minimal type (ie of U-rank 1); if it is msa then X can be thought of as the canonical base.]

DEFINITION 4.17. Let μ be a function from the set of isomorphism types of msa extensions to the natural numbers.

Let $\bar{\mathcal{C}}_\mu$ be consist of structures $C \in \bar{\mathcal{C}}_0$ with the following property for each msa $X \leq Y$: Suppose $X', Y'_1, \ldots, Y'_n \subseteq C$ are such that $Y'_i \cap Y'_j = X$ (for $i \neq j$) and there are isomorphisms $Y \to Y'_i$ which all send X to X' in the same way. Then $n \leq \mu(X,Y)$. Let $\mathcal{C}_\mu$ be the finite structures in $\bar{\mathcal{C}}_\mu$.

So if $C \in \bar{\mathcal{C}}_\mu$ there are at most $\mu(X,Y)$ (disjoint) copies of Y over X, whenever $X \subseteq C$ and $X \leq Y$ is msa. In particular, if b is in one of these copies, then b is in the algebraic closure of X. Thus, by Remarks 4.16:

LEMMA 4.18. *Suppose $C \in \bar{\mathcal{C}}_\mu$ and $B \subseteq C$. If $d_B(a/B) = 0$ then a is algebraic over B.*

THEOREM 4.19 (Amalgamation Lemma). *Suppose that for every msa $X \leq Y$ we have $\mu(X,Y) \geq \delta(X)$. Then $(\bar{\mathcal{C}}, \mu)$ is an amalgamation class: if $B_1, B_2 \in \bar{\mathcal{C}}_\mu$ and $f_i : A \to B_i$ are $\leq$-embeddings, then there exists $C \in \bar{\mathcal{C}}_\mu$ and $\leq$-embeddings $g_i : B_i \to C$ with $g_1 \circ f_1 = g_2 \circ f_2$.*

Of course, we cannot necessarily take C here to be the free amalgam of B_1 and B_2 over A. However, using the same proof as before, we have:

THEOREM 4.20 (The generic structure for $(\mathcal{C}_\mu; \leq)$). *Suppose $\mu(X,Y) \geq \delta(X)$ for all msa $X \leq Y$. Then there is a countable $\mathcal{M}_\mu \in \bar{\mathcal{C}}_\mu$ satisfying the following properties:*

($C1(\mu)$): *$\mathcal{M}_\mu$ is the union of a chain of finite substructures $B_1 \leq B_2 \leq B_3 \leq \ldots$ all of which are in $\mathcal{C}_\mu$.*

($C2(\mu)$): *If $A \leq \mathcal{M}_\mu$ is finite and $A \leq B \in \mathcal{C}_\mu$, then there is an embedding $f : B \to \mathcal{M}_\mu$ with $f(B) \leq \mathcal{M}_\mu$ and which is the identity on A.*

Moreover $\mathcal{M}_\mu$ is uniquely determined up to isomorphism by these two properties and is $\leq$-homogeneous (meaning: any isomorphism between finite self-sufficient substructures of $\mathcal{M}_\mu$ extends to an automorphism of $\mathcal{M}_\mu$).

THEOREM 4.21. *With the above notation, $Th(\mathcal{M}_\mu)$ is strongly minimal.*

It is easy to see (using $C2(\mu)$) that if $A \subseteq \mathcal{M}_\mu$ is finite, and $b, b' \in \mathcal{M}_\mu$ are such that $d(b/A) = d(b'/A)$, then there is an automorphism of $\mathcal{M}_\mu$ which fixes A and sends b to b'. It follows that $\mathrm{tp}(b/\mathrm{acl}(A)) = \mathrm{tp}(b'/\mathrm{acl}(A))$. However, we cannot immediately conclude from this that there is a unique non-algebraic 1-type over A: we do not know *a priori* that $\mathcal{M}_\mu$ is ω-saturated.

As in our analysis of $Th(\mathcal{M})$, in order to prove Theorem 4.21, we need a stronger version of the amalgamation property.

THEOREM 4.22 (Strong algebraic amalgamation property). *Suppose $\mu(X,Y) \geq \delta(X)$ for all msa $X \leq Y$. Suppose $A \subseteq B_1, B_2 \in \mathcal{C}_\mu$ and $A \leq B_1$ is simply algebraic. Let E be the free amalgam of B_1 and B_2 over A. Then $B \in \mathcal{C}_\mu$ unless one of the following holds:*

(1) There is some $X \subseteq A$ such that $X \leq Y = X \cup (B_1 \setminus A)$ is msa, and B_2 contains $\mu(X,Y)$ copies of Y which are pairwise disjoint over X.

(2) There is a set $Z \subseteq B_2$ such that $Z \cap A \not\leq Z$ and B_1 contains a copy of Z.

Note that if $A \leq B$, then (2) cannot occur. The amalgamation lemma 4.19 follows fairly easily from this (in case (1) we can amalgamate by identifying $B_1 \setminus A$ with one of the copies of $Y \setminus X$ over X, noting that as $B_1 \in \mathcal{C}_\mu$ at least one of these must be in B_2).

We now write down axioms T_μ whose models $\mathcal{N}$ satisfy:

(A1) $\mathcal{N} \in \bar{\mathcal{C}}_\mu$;

(A2) For each n, m there is a set of size n with no relations on it which is $\leq^m$ in $\mathcal{N}$;

(A3) Suppose $A \subseteq \mathcal{N}$ and $A \leq B$ is simply algebraic. Suppose furthermore that whenever $Z \subseteq \mathcal{M}$ has an isomorphic copy in B, then $A \cap Z \leq Z$. THEN there are $\mu(X,Y)$ copies of B over A in $\mathcal{N}$ which are pairwise disjoint over A.

THEOREM 4.23. *We have $\mathcal{M}_\mu \models T_\mu$ and $\mathcal{M}_\mu$ is ω-saturated. Thus $\mathcal{M}_\mu$ is strongly minimal.*

PROOF. (Sketch)

Step 1: $\mathcal{M}_\mu \models T_\mu$. A1, A2 are easy. To see that axioms A3 hold, use Theorem 4.22.

Step 2: If $\mathcal{N}, \mathcal{N}'$ are models of T_μ of infinite dimension, then the set of isomorphisms between finite $\leq$-substructures is a back-and-forth system between $\mathcal{N}$ and $\mathcal{N}'$.

Step 3: By step 2, as $\mathcal{M}_\mu$ is infinite dimensional, it is isomorphic to any countable elementary extension of itself. It follows that $\mathcal{M}_\mu$ is ω-saturated.

Step 4: $\mathcal{M}_\mu$ is minimal and therefore, by ω-saturation, it is strongly minimal. Indeed, suppose $\phi(x, \bar{a})$ is non-algebraic. By a compactness argument and ω-saturation, there is $b \in \mathcal{M}_\mu$ with $\mathcal{M}_\mu \models \phi(b, \bar{a})$ and $b \notin \mathrm{acl}(\bar{a})$. If $\neg\phi(x, \bar{a})$ is non-algebraic then similarly, there is $b' \in \mathcal{M}_\mu$ with $\mathcal{M}_\mu \models \phi(b', \bar{a})$ and $b' \notin \mathrm{acl}(A)$. But then, as we already noted, $\mathrm{tp}(b/\bar{a}) = \mathrm{tp}(b'/\bar{a})$ which is impossible. □

Exercises I

LMS/EPSRC Short Course on Model Theory, Leeds, July 18-23, 2010.
Exercises on David Evans' lectures 1 and 2.

A selection of problems of varying difficulty: choose something appropriate to your background. Some of them are just something to think about. Do some of question 5.

1. (See the description of the Stone space topology in Remark 1.2.) Suppose T is a complete L-theory. Use the Compactness Theorem to prove that $S_n(T)$ is a compact topological space.

2. A type $p(\bar{x}) \in S_n(T)$ is *isolated* if there is a formula $\theta(\bar{x}) \in p(\bar{x})$ such that $[\theta(\bar{x})] = \{p(\bar{x})\}$.

(i) Compare this with the definition of an isolated point in a topological space.

(ii) Show that $p(\bar{x})$ is isolated by $\theta(\bar{x})$ iff for every formula $\phi(\bar{x})$ and model $\mathcal{M}$ of T, either $\mathcal{M} \models (\forall \bar{x})(\theta(\bar{x}) \rightarrow \phi(\bar{x}))$ or $\mathcal{M} \models (\forall \bar{x})(\theta(\bar{x}) \rightarrow \neg\phi(\bar{x}))$.

(iii) Prove that if $p(\bar{x})$ is isolated and $\mathcal{M}$ is any model of T, there is $\bar{a} \in \mathcal{M}$ which realises $p(\bar{x})$.

3. Suppose $\mathcal{M}_1$, $\mathcal{M}_2$ are countable, ω-saturated models of a complete theory T. Prove that $\mathcal{M}_1$ and $\mathcal{M}_2$ are isomorphic.

4. Prove Theorem 1.9.

5. In each of the following, a language L (with equality) is given and a theory T is described. In each case, write down axioms for T and prove completeness and quantifier elimination of T.

(i) L has a single binary relation symbol E and T says that E is an equivalence relation with infinitely many classes and all classes are infinite.

(ii) L just has $=$; the models of T are the infinite sets.

(iii) F is a fixed field and $L = (+, -, 0, (f_\alpha : \alpha \in F))$; T is the theory of infinite vector spaces in this language (with $+, -, 0$ the operations and zero in the vector space , and f_α is the unary function: scalar multiplication by α).

(iv) L has a 2-ary relation symbol $\leq$ and the models of T are the dense linear orders without endpoints.

(v) L has a 2-ary relation symbol $\leq$ and a 1-ary predicate $P(x)$. The models of T are the dense linear orders without endpoints in which $P(x)$ picks out a subset which is dense and whose complement is dense.

6. Let L be the language $(+, -, 0)$. Let G be the abelian group $(\mathbb{Z}; +, -, 0)$ regarded as an L-structure. Prove that $Th(G)$ does not have quantifier elimination. Try to write down axioms for $Th(G)$.

7. Use the Ryll-Nardzewski Theorem (2.1) to show that if $\mathcal{M}$ is ω-categorical then algebraic closure is locally finite: if A is a finite subset of $\mathcal{M}$ then $\operatorname{acl}(A)$ is finite.

8. If T is one of the theories in question 5 and $A \subseteq \mathcal{M} \models T$, describe the complete types in $S_1(A)$ (for example, for $A = \emptyset$, or A infinite). Express $|S_1(A)|$ in terms of $|A|$.

9. Using 5(iii) prove that an infinite vector space (thought of as an L-structure in the given way) is strongly minimal. Show that algebraic closure is the same as linear closure (and therefore dimension is linear dimension).

10. Suppose $(X; \mathrm{cl})$ is a pregeometry and Z is a basis of X. Prove that $\mathrm{cl}(Z) = X$. Show that any two bases of X have the same cardinality. [First show that $a_1, \ldots, a_n \in \mathrm{cl}(b_1, \ldots, b_m)$ are independent, then $n \leq m$.]

Exercises II

LMS/EPSRC Short Course on Model Theory, Leeds, July 18-23, 2010.
Exercises on David Evans' lectures for the Tuesday afternoon session.

1. Show that if T is totally transcendental then there is a model $\mathcal{M} \models T$ and a strongly minimal formula $\phi(x, \bar{a})$ with parameters in $\mathcal{M}$.
Variation: Use the same argument to show that if $\mathcal{M} \models T$ there is an $L(M)$-formula $\phi(x, \bar{a})$ which is *minimal* in $\mathcal{M}$ (meaning: for every $L(M)$-formula $\psi(x)$ either $\phi(x, \bar{a}) \wedge \psi(x)$ or $\phi(x, \bar{a}) \wedge \neg\psi(x)$ is algebraic).

2. Suppose $\mathcal{M}$ is an L-structure and $\psi(x, \bar{y})$ an L-formula such that for each $n < \omega$ there is a tuple $\bar{a}_n \in \mathcal{M}$ with the property that $\psi(x, \bar{a}_n)$ is algebraic and has $\geq n$ solutions (in $\mathcal{M}$). Prove that $Th(\mathcal{M})$ has a Vaughtian pair.

The following use the notation of Section 4 of the notes.

3. Give an example of $A \leq B \in \mathcal{C}_0$ with $\delta(A) = \delta(B) = 3$, $|B \setminus A| = 3$ and if $A \subset B' \subset B$, then $\delta(B') > \delta(A)$.
(Harder) Do this with $|B \setminus A| = n$.

4. Give an example of $B \in \mathcal{C}_0$ where $\mathrm{cl}_B^{\leq}$ does not satisfy the exchange property.

5. (Easy) Give an example of $B \in \mathcal{C}_0$ and $a, b \in B$ such that $\mathrm{cl}_B^d(\{a, b\}) \neq \mathrm{cl}_B^d(a) \cup \mathrm{cl}_B^d(b)$.

6. (Harder) Let B be a finite set and R a set of subsets of size 3 of B. Consider the structure $(B; R)$. Show that $B \in \mathcal{C}_0$ iff there is an injective function $t : R \to B$ with $t(r) \in r$ for each $r \in R$.
[$\Leftarrow$ is fairly easy; $\Rightarrow$ uses Hall's Marriage Theorem.]

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