Solid Mechanics: Linear Elasticity

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1 Introduction

In a solid material (e.g. steel, rubber, wood, crystals, etc.) a **deformation** (i.e. a change of shape) can be sustained only if a system of forces (loads) is applied to the bounding surfaces, so setting up internally a distribution of **stress** (i.e. force per unit area). The defining characteristic of an elastic material is that when these loads are removed the body returns **exactly** to its original shape. The original state in which no loads are applied is often called the **natural state**, or **reference configuration**.

Almost all engineering materials possess the property of elasticity provided the external loads are not too large. If the loads are increased beyond a certain limit, different for each material, then the material will **fail**, either by **fracture** or by **flow**. In neither case does the material return to its original shape when the loads are removed and we say that the **elastic limit** has been exceeded. A solid material that fails by fracture is said to be **brittle** and one that fails by flow is said to be **plastic**.

We do not consider atomic structure. We assume that matter is homogeneous and continuously distributed over the material body, so that the smallest part of the body possesses the same physical properties as the whole body. The theory of the mechanical behaviour of such materials is called **continuum mechanics**. We consider therefore only the **macroscopic** (large scale) behaviour of materials, which is adequate for most engineering purposes, and ignore the **microscopic** (small scale) behaviour.

We further assume that the deformation is small so that the elastic material is **linearly elastic**. For almost all engineering materials the linear theory of elasticity holds if the applied loads are small enough.

This unit discusses only the linear theory of elasticity.

2 The kinematics of deformation: strain

Kinematics is the study of motion (and deformation) without regard for the forces causing it.

A material body $\mathcal{B}$ occupies a region B_0 of space at time $t = 0$, its **reference configuration**, and a **material particle** has position vector $\mathbf{X}$, with Cartesian components X_i, $i = 1, 2, 3$, with respect to the orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$:

$$\mathbf{X} = \sum_{i=1}^{3} X_i\, \mathbf{e}_i.$$

Figure 1: Reference configuration   Current configuration

The body undergoes a **motion**, or **deformation**, during which the material particle at $\mathbf{X}$ when $t = 0$ is moved to

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t) \tag{2.1}$$

at time t and then has position vector

$$\mathbf{x} = \sum_{i=1}^{3} x_i \, \mathbf{e}_i,$$

with Cartesian components x_i, $i = 1, 2, 3$. Using the **Einstein summation convention**, by which twice-occurring subscripts are summed over (from 1 to 3), we can write

$$\mathbf{X} = X_i \, \mathbf{e}_i, \quad \mathbf{x} = x_i \, \mathbf{e}_i.$$

We write

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{X}, t), \quad x_i = X_i + u_i(\mathbf{X}, t), \quad i = 1, 2, 3, \tag{2.2}$$

so that $\mathbf{u}(\mathbf{X}, t)$ is the **particle displacement** with components $u_i, i = 1, 2, 3$. The **displacement gradient**, $\mathbf{h}$, is the 3×3 matrix with components

$$h_{ij} = \frac{\partial u_i}{\partial X_j}.$$

In the **linear theory of elasticity** we assume not only that the components u_i are small but also that each of the derivatives h_{ij} is small. This has an important consequence: since $\mathbf{x} - \mathbf{X} = \mathbf{u}$ is small we can afford to replace coordinates $\mathbf{X}$ by $\mathbf{x}$ and write in place of (2.2)

$$\mathbf{x} = \mathbf{X} + \mathbf{u}(\mathbf{x}, t), \quad x_i = X_i + u_i(\mathbf{x}, t), \quad i = 1, 2, 3, \tag{2.3}$$

and work in future only with the coordinates x_i and the particle displacements $u_i(\mathbf{x}, t)$.

The **particle velocity** is

$$\mathbf{v} = \left.\frac{\partial \mathbf{x}}{\partial t}\right|_{\mathbf{X}} \approx \left.\frac{\partial \mathbf{u}}{\partial t}\right|_{\mathbf{x}},$$

and the **particle acceleration** is similarly approximated by

$$\mathbf{a} = \left.\frac{\partial^2 \mathbf{x}}{\partial t^2}\right|_{\mathbf{X}} \approx \left.\frac{\partial^2 \mathbf{u}}{\partial t^2}\right|_{\mathbf{x}}.$$

The deformation enters into the linear theory of elasticity only through the linear **strain tensor**, $\mathbf{e}$, which has components

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.4}$$

$(i, j = 1, 2, 3)$. For example,

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1} \quad \text{and} \quad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right).$$

A **tensor** is a linear map from one vector space to another; but all such maps can be represented by **matrices**. So, for **tensor**, think **matrix**.

The strain tensor $\mathbf{e}$ is **symmetric**:

$$\mathbf{e}^T = \mathbf{e}, \quad e_{ji} = e_{ij},$$

i.e. $e_{12} = e_{21},\ e_{23} = e_{32},\ e_{31} = e_{13}$. We see this from (2.4):

$$e_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = e_{ij}.$$

The matrix (e_{ij}) of strain components is therefore real and symmetric and so, by the theory of linear algebra, has 3 real eigenvalues with corresponding unit eigenvectors which are mutually orthogonal. The eigenvalues are known as the **principal strains** and the unit eigenvectors are the **principal axes of strain**.

Triaxial stretch

Consider a unit cube aligned with the coordinate axes and subjected to the deformation

$$u_1 = e_1 x_1, \quad u_2 = e_2 x_2, \quad u_3 = e_3 x_3, \tag{2.5}$$

independent of time t, in which e_i are the constant (small) strains. Note that $e_i > 0$ corresponds to a **stretch**, and $e_i < 0$ to a **contraction**.

From (2.5) we see that the strain components are given by

$$e_{11} = \frac{\partial u_1}{\partial x_1} = e_1, \quad e_{22} = \frac{\partial u_2}{\partial x_2} = e_2, \quad e_{33} = \frac{\partial u_3}{\partial x_3} = e_3,$$

with all the other components vanishing: $e_{12} = e_{21} = e_{23} = e_{32} = e_{31} = e_{13} = 0$. Thus

$$(e_{ij}) = \begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

Figure 2: Triaxial stretch

and we see that the principal strains are given by $e_1,\ e_2,\ e_3$. Also, the principal axes of strain are aligned with the coordinate axes of Figure 2 and have components

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

respectively.

After the triaxial stretch the new volume is

$$(1+e_1)(1+e_2)(1+e_3) = 1 + e_1 + e_2 + e_3 + e_1e_2 + e_2e_3 + e_3e_1 + e_1e_2e_3.$$

Since the e_i are small, the relative change in volume, known as the **dilatation**, is

$$\Delta = e_1 + e_2 + e_3 = \frac{\text{change in volume}}{\text{original volume}}.$$

But in this case $e_1 = e_{11}$, $e_2 = e_{22}$, $e_3 = e_{33}$, so

$$\Delta = e_{11} + e_{22} + e_{33} = \operatorname{tr}\mathbf{e} = e_{kk}, \tag{2.6}$$

where k is summed over. In fact, for any deformation, with $\mathbf{e}$ not necessarily diagonal, the dilatation is given by (2.6). To see this we observe that any strain tensor $\mathbf{e}$ can be diagonalized (because symmetric) by a suitable rotation of coordinates and then in the new coordinates the deformation is necessarily of the form (2.5). So the dilatation is given by (2.6). But $\operatorname{tr}\mathbf{e} = e_{kk}$ is an **invariant** quantity, unchanged by any rotation of coordinates. Therefore (2.6) represents the dilatation for general $\mathbf{e}$.

Conservation of mass

The mass m of the cube and the cuboid in Figure 2 are the same. Let ρ_0 and ρ be the initial and final densities:

$$\rho_0 = \frac{m}{1}, \quad \rho = \frac{m}{1+\Delta} \quad \Rightarrow \quad \rho = \rho_0(1+\Delta)^{-1}.$$

So, by the binomial theorem, with Δ small, the **conservation of mass** reads

$$\rho = \rho_0(1-\Delta). \tag{2.7}$$

If the volume increases ($\Delta > 0$), the density decreases ($\rho < \rho_0$);
if the volume decreases ($\Delta < 0$), the density increases ($\rho > \rho_0$).

Simple shear

Consider the same unit cube aligned with the coordinate axes but subjected to the shear deformation

$$u_1 = \gamma x_2, \quad u_2 = 0, \quad u_3 = 0, \tag{2.8}$$

with γ a positive constant.

Figure 3: Simple shear

Then $\dfrac{\partial u_1}{\partial x_2} = \gamma$, with all other components of deformation gradient vanishing. So

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(\gamma + 0) = \frac{\gamma}{2} = e_{21},$$

with all other strain components vanishing:

$$(e_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For general strain tensor $\mathbf{e}$ the diagonal components e_{11}, e_{22}, e_{33} are termed the **normal strains**, whereas the off-diagonal elements $e_{12} = e_{21}$, $e_{23} = e_{32}$, $e_{31} = e_{13}$, are termed the **shear strains**.

3 The theory of stress: equations of motion

The forces acting on a deformed body are of two kinds.

1. **Body force**, $\mathbf{b}$, measured per unit mass, acts on volume elements, e.g. gravity, inertial forces.

2. **Contact force**, $\mathbf{t}(\mathbf{n})$, measured per unit area.

Figure 4: Contact forces

Take an arbitrary surface element da with unit normal vector $\mathbf{n}$. The material on the side of da **into** which $\mathbf{n}$ points exerts a force $\mathbf{t}(\mathbf{n})da$, across the surface element da, on the material on the other side of da. The vector $\mathbf{t}(\mathbf{n})$ is the **traction vector**.

Example 1: Uniaxial tension A force per unit (deformed) area T is applied to a

Figure 5: Uniaxial tension

cuboid in the x_1-direction causing extension in the x_1-direction and (usually) lateral contraction. Now

$$\mathbf{t}(\mathbf{e}_1) = T\mathbf{e}_1, \quad \text{but} \quad \mathbf{t}(\mathbf{e}_2) = \mathbf{0},$$

so $\mathbf{t}(\mathbf{n})$ depends on $\mathbf{n}$, even for fixed $\mathbf{x}$ and t.

Figure 6: Simple shear

Example 2: Simple shear In this case $\mathbf{t}(\mathbf{e}_2) = T\mathbf{e}_1$, so $\mathbf{t}(\mathbf{n})$ is not necessarily parallel to $\mathbf{n}$.

Consider a cuboid of deformed material with sides δx_1, δx_2, δx_3 parallel to the coordinate axes. The tractions on each face are as shown.

Figure 7: The stress components

The traction vector $\mathbf{t}(\mathbf{e}_1)$ on the face with normal $\mathbf{e}_1$ has components $\begin{pmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{pmatrix}$,

the traction vector $\mathbf{t}(\mathbf{e}_2)$ on the face with normal $\mathbf{e}_2$ has components $\begin{pmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{pmatrix}$,

the traction vector $\mathbf{t}(\mathbf{e}_3)$ on the face with normal $\mathbf{e}_3$ has components $\begin{pmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{pmatrix}$.

These column vectors are put together to form the 3×3 matrix of components of the **stress tensor**, $\boldsymbol{\sigma}$:

$$(\sigma_{ij}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}. \tag{3.1}$$

The diagonal elements σ_{11}, σ_{22}, σ_{33} are termed **normal stresses** and the off-diagonal elements are termed **shear stresses**.

It can be shown that for arbitrary unit normal vector $\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3 = n_i\mathbf{e}_i$, the traction vector is given by

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n}, \quad t_i = \sigma_{ij}n_j, \quad \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}, \tag{3.2}$$

so the traction components t_i are a linear combination of the unit surface normal components n_i.

The normal component of traction is

$$t_n = \mathbf{t}(\mathbf{n}) \cdot \mathbf{n} = t_i n_i = \sigma_{ij} n_j n_i = \sigma_{ij} n_i n_j, \tag{3.3}$$

from (3.2). The **normal stress** t_n is **tensile** when positive and **compressive** when negative.

Figure 8: Normal and shear components of traction

The tangential component of traction is

$$\mathbf{t}(\mathbf{n}) - t_n \mathbf{n}$$

with magnitude

$$\tau_n = |\mathbf{t}(\mathbf{n}) - t_n \mathbf{n}| = \{|\mathbf{t}(\mathbf{n})|^2 - t_n^2\}^{\frac{1}{2}}, \tag{3.4}$$

called the **shear stress**.

Example 3 For the stress components at a point P

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{pmatrix}$$

(a) find the traction components at P on a plane whose outward unit normal has components $(\frac{3}{5},\ 0,\ \frac{4}{5})$,
(b) find the normal and shear stresses at P on the given plane.

Solution (a) The traction components are

$$\mathbf{t}(\mathbf{n}) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ \frac{13}{5} \end{pmatrix}$$

(b) From (3.3) the normal stress is

$$t_n = \mathbf{n} \cdot \mathbf{t}(\mathbf{n}) = \left(\frac{3}{5},\ 0,\ \frac{4}{5}\right) \cdot \begin{pmatrix} 3 \\ 6 \\ \frac{13}{5} \end{pmatrix} = \frac{97}{25}.$$

From (3.4) the shear stress is

$$\tau_n = |\mathbf{t}(\mathbf{n}) - t_n \mathbf{n}| = \{|\mathbf{t}(\mathbf{n})|^2 - t_n^2\}^{\frac{1}{2}} = \left\{3^2 + 6^2 + \left(\frac{13}{5}\right)^2 - \left(\frac{97}{25}\right)^2\right\}^{\frac{1}{2}} = \frac{\sqrt{22,941}}{25} \approx 6.059.$$

The equations of motion

The balance of linear momentum

Figure 9: Balance of forces in the x_2-direction

Consider an elementary cuboid of material with sides δx_1, δx_2, δx_3 parallel to the coordinate axes. The cuboid has volume $\delta v = \delta x_1 \delta x_2 \delta x_3$. We calculate all the forces acting on the element in the x_2-direction. Unbalanced normal stress $\dfrac{\partial \sigma_{22}}{\partial x_2}\delta x_2$ acts on an area $\delta x_1 \delta x_3$, giving a normal force on the element in the x_2-direction of $\dfrac{\partial \sigma_{22}}{\partial x_2}\delta x_2 \times \delta x_1 \delta x_3$ $= \dfrac{\partial \sigma_{22}}{\partial x_2}\delta v$. Unbalanced shear stresses $\dfrac{\partial \sigma_{23}}{\partial x_3}\delta x_3$ and $\dfrac{\partial \sigma_{21}}{\partial x_1}\delta x_1$ act on areas $\delta x_1 \delta x_2$ and $\delta x_2 \delta x_3$, respectively, giving shear forces in the x_2-direction of $\dfrac{\partial \sigma_{23}}{\partial x_3}\delta v$ and $\dfrac{\partial \sigma_{21}}{\partial x_1}\delta v$, respectively. The mass of the element is $\rho \delta v$ and so the component of body force (per unit mass) b_2 in the x_2-direction contributes a force $\rho b_2 \delta v$ on the element in that direction.

By Newton's second law the total of these forces can be equated to the rate of change of linear momentum in the x_2-direction:

$$\frac{\partial \sigma_{21}}{\partial x_1}\delta v + \frac{\partial \sigma_{22}}{\partial x_2}\delta v + \frac{\partial \sigma_{23}}{\partial x_3}\delta v + \rho b_2 \delta v = \rho \delta v \frac{\partial^2 u_2}{\partial t^2}.$$

On dividing by δv we obtain the equation of motion

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = \rho \frac{\partial^2 u_2}{\partial t^2}.$$

By similar arguments in the x_1- and x_3-directions we deduce a complete set of three **equations of motion:**

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \rho b_1 = \rho \frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + \rho b_2 = \rho \frac{\partial^2 u_2}{\partial t^2}, \tag{3.5}$$

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + \rho b_3 = \rho \frac{\partial^2 u_3}{\partial t^2}.$$

These equations are conveniently written in suffix notation as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{3.6}$$

for each $i = 1, 2, 3$, and summing over j.

In the static case, in which no quantities vary with time t then (3.6) reduce to the **equilibrium equations**

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = 0. \tag{3.7}$$

The balance of angular momentum

Figure 10: Shear stresses

Taking moments about the origin O shows that

$$\sigma_{21} = \sigma_{12}.$$

Similarly, we obtain

$$\sigma_{31} = \sigma_{13}, \quad \sigma_{23} = \sigma_{32}.$$

So the balance of angular momentum reduces to the
symmetry of the stress tensor:

$$\boldsymbol{\sigma}^T = \boldsymbol{\sigma}, \quad \sigma_{ji} = \sigma_{ij}. \tag{3.8}$$

(Recall also the symmetry of the strain tensor $\mathbf{e}$).

Properties of the stress tensor, $\boldsymbol{\sigma}$

It follows from this symmetry that $\boldsymbol{\sigma}$ has three real eigenvalues with corresponding real unit eigenvectors. These three eigenvalues are termed the **principal stresses** and the corresponding unit eigenvectors are termed the **principal axes of stress**. They are mutually orthogonal.

If the stress has diagonal form

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

then the principal stresses are $\sigma_1, \sigma_2, \sigma_3$ with corresponding principal axes of stress $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, along the coordinate axes. The traction on a plane with normal components (n_1, n_2, n_3) is

$$\mathbf{t}(\mathbf{n}) = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} \sigma_1 n_1 \\ \sigma_2 n_2 \\ \sigma_3 n_3 \end{pmatrix}.$$

The normal stress is therefore, from (3.3),

$$t_n = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \tag{3.9}$$

and the squared shear stress can, from (3.4), be shown to be given by

$$\tau_n^2 = (\sigma_1 - \sigma_2)^2 n_1^2 n_2^2 + (\sigma_1 - \sigma_3)^2 n_1^2 n_3^2 + (\sigma_2 - \sigma_3)^2 n_2^2 n_3^2. \tag{3.10}$$

If the stress takes the form

$$\boldsymbol{\sigma} = -p\mathbf{I}, \quad \sigma_{ij} = -p\delta_{ij}$$

in which δ_{ij} denotes the **Kronecker delta**

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$$

in other words the components of the unit tensor $\mathbf{I}$, it is said to be **spherical** (or **hydrostatic**) and p is the **pressure**. For a surface segment with unit normal $\mathbf{n}$

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n} = -p\mathbf{I}\mathbf{n} = -p\mathbf{n}, \qquad t_i = -pn_i,$$

so that the normal stress and shear stress are, from (3.3) and (3.4),

$$t_n = (-p\mathbf{n}) \cdot \mathbf{n} = -p\mathbf{n} \cdot \mathbf{n} = -p \quad \text{and} \quad \tau_n = 0.$$

If the stress is spherical, therefore, the traction is purely normal on every surface element, so that there is no shear stress on any surface element.

Example 4 For the matrix of stress components of **Example 3**, namely,

$$(\sigma_{ij}) = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{pmatrix},$$

find the principal stresses and the corresponding principal axes of stress.

Solution Let σ denote the eigenvalues and $\mathbf{n}$ the eigenvectors of the stress tensor $\boldsymbol{\sigma}$. Then, by definition, $\boldsymbol{\sigma}\mathbf{n} = \sigma\mathbf{n}$, so that $(\boldsymbol{\sigma} - \sigma\mathbf{I})\mathbf{n} = \mathbf{0}$, or, in components,

$$\begin{pmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

There are non-trivial solutions (i.e. $\mathbf{n} \neq \mathbf{0}$) if and only if the determinant of coefficients vanishes:

$$\begin{vmatrix} \sigma_{11}-\sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma \end{vmatrix} = 0.$$

In the present example this condition becomes

$$\begin{vmatrix} 1-\sigma & 2 & 3 \\ 2 & 4-\sigma & 6 \\ 3 & 6 & 1-\sigma \end{vmatrix} = 0.$$

Expand by first row:

$$(1-\sigma)\{(4-\sigma)(1-\sigma) - 36\} - 2\{2(1-\sigma) - 18\} + 3\{12 - 3(4-\sigma)\} = 0.$$

Simplify the curly brackets:

$$(1-\sigma)(\sigma^2 - 5\sigma - 32) - 2(-2\sigma - 16) + 9\sigma = 0.$$

Multiply out and collect terms:

$$-\sigma^3 + 6\sigma^2 + 40\sigma = 0.$$

Change overall sign and observe that $\sigma = 0$ is a root:

$$\sigma(\sigma^2 - 6\sigma - 40) = 0, \qquad \text{and factorise:} \quad \sigma(\sigma - 10)(\sigma + 4) = 0.$$

Therefore the principal stresses are

$$\sigma_1 = 0, \quad \sigma_2 = 10, \quad \sigma_3 = -4.$$

We must now find the corresponding unit eigenvectors.
For $\sigma = \sigma_1 = 0$ we must find components n_i such that

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

2nd equation same as 1st. 1st and 3rd give $n_3 = 0$, and then $n_1 = 2$, $n_2 = -1$ are possible solutions. The unit vector $\mathbf{n}_1$ therefore has components $5^{-\frac{1}{2}}(2, -1, 0)$.
For $\sigma = \sigma_2 = 10$ we find the unit vector $\mathbf{n}_2$ has components $70^{-\frac{1}{2}}(3, 6, 5)$.
For $\sigma = \sigma_3 = -4$ we find the unit vector $\mathbf{n}_3$ has components $14^{-\frac{1}{2}}(1, 2, -3)$.

4 Linear isotropic elasticity: constitutive equations and homogeneous deformations

The constitutive equation of isotropic linear elasticity (the **stress-strain law**) is

$$\sigma_{ij} = \lambda e_{pp}\delta_{ij} + 2\mu e_{ij}. \tag{4.1}$$

known as the **generalized Hooke's law**, with e_{ij} defined by (2.4). The material constants (or **elastic moduli**) λ and μ are known as the **Lamé moduli** and must be measured experimentally for each material.

In the absence of body force the equations of equilibrium (3.7) are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0. \tag{4.2}$$

Homogeneous deformations

A deformation is said to be **homogeneous** if the strain tensor $\mathbf{e}$ is independent of $\mathbf{x}$, i.e. spatially uniform. From (4.1) the stress tensor $\boldsymbol{\sigma}$ is also uniform in space. Thus (4.2) is satisfied trivially. Therefore, any homogeneous deformation satisfies the equations of equilibrium and so is possible in an elastic material, in the absence of body force.

We examine various examples of homogeneous deformation.

(a) Uniform dilatation

An isotropic elastic sphere of radius a centred on the origin undergoes the uniform dilatation

$$u_1 = \alpha x_1, \quad u_2 = \alpha x_2, \quad u_3 = \alpha x_3, \quad (\text{or} \quad \mathbf{u} = \alpha\mathbf{x}), \tag{4.3}$$

where α is a dimensionless constant; $\alpha > 0$ for expansion and $\alpha < 0$ for contraction. Now

$$\left(\frac{\partial u_i}{\partial x_j}\right) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

and so

$$e_{ij} = \alpha\delta_{ij}.$$

The dilatation is $\Delta = e_{pp} = 3\alpha$, and from (4.1),

$$\sigma_{ij} = 3\alpha\lambda\delta_{ij} + 2\mu\alpha\delta_{ij} = \alpha(3\lambda + 2\mu)\delta_{ij}$$

so $\sigma_{11} = \sigma_{22} = \sigma_{33} = -p$, where $p = -\alpha(3\lambda + 2\mu)$, with all shear stresses vanishing. We have

$$p = -K\Delta$$

where

$$K := \frac{\frac{1}{3}\sigma_{pp}}{e_{pp}}, \quad \text{or} \quad K = \frac{-p}{\Delta} = \lambda + \frac{2}{3}\mu \tag{4.4}$$

is the **bulk modulus** of the material.

(b) Simple shear

The displacements are

$$u_1 = \gamma x_2, \qquad u_2 = 0, \qquad u_3 = 0,$$

so that

$$(e_{ij}) = \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad e_{pp} = 0.$$

So from (4.1),

Figure 11: Simple shear

$$\sigma_{ij} = 2\mu e_{ij} = \begin{pmatrix} 0 & \gamma\mu & 0 \\ \gamma\mu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $\sigma_{12} = \gamma\mu, \quad e_{12} = \frac{1}{2}\gamma$ are the only non-zero stresses and strains.
The quantity

$$\mu = \frac{\sigma_{12}}{2e_{12}}$$

is known as the **shear modulus** of the material.

(c) Simple extension (uniaxial tension)

Suppose that a cylindrical rod lies with its generators parallel to the x_1-direction and suffers a uniform tension T at each end:

$$(\sigma_{ij}) = \begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We assume a displacement

$$u_1 = \alpha x_1, \qquad u_2 = -\beta x_2, \qquad u_3 = -\beta x_3,$$

where $\alpha > 0$ corresponds to extension of the rod and $\beta > 0$ to lateral contraction. So

$$(e_{ij}) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\beta & 0 \\ 0 & 0 & -\beta \end{pmatrix}, \qquad e_{pp} = \alpha - 2\beta.$$

Figure 12: Uniaxial tension

From (4.1)

$$\sigma_{11} = T = \lambda(\alpha - 2\beta) + 2\mu\alpha, \quad \sigma_{22} = \sigma_{33} = 0 = \lambda(\alpha - 2\beta) - 2\mu\beta.$$

From the second equation we get β and then from the first we get T:

$$\beta = \frac{\alpha\lambda}{2(\lambda + \mu)}, \qquad T = \frac{\alpha\mu(3\lambda + 2\mu)}{\lambda + \mu}.$$

We define the **Young's modulus** E by

$$E := \frac{\sigma_{11}}{e_{11}}, \quad \text{so} \quad E = \frac{T}{\alpha} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \tag{4.5}$$

and **Poisson's ratio** ν by

$$\nu := \frac{-e_{22}}{e_{11}}, \quad \text{so} \quad \nu = \frac{\beta}{\alpha} = \frac{\lambda}{2(\lambda + \mu)}, \tag{4.6}$$

both in terms of λ and μ. E is the tension per unit axial extension and ν is the transverse contraction per unit axial extension.

Elastic constants

We have introduced elastic constants K, E, ν in addition to the Lamé constants λ, μ and have provided clear mechanical interpretations of K, E, ν and μ. Any three of λ, μ, K, E, ν can be expressed in terms of the other two by manipulating (4.4), (4.5) and (4.6).

We shall suppose, on physical grounds, that

$$K > 0, \qquad \mu > 0 : \tag{4.7}$$

$K > 0$	implies that a compressive pressure produces a decrease in volume
$\mu > 0$	implies a shear stress produces shear strain in the same direction, not in the opposite direction.

By manipulating (4.4)–(4.6) we obtain

$$E = \frac{9K\mu}{3K + \mu}, \qquad \nu = \frac{3K - 2\mu}{2(3K + \mu)} \tag{4.8}$$

and so inequalities (4.7) imply that

$$E > 0, \qquad -1 < \nu < \frac{1}{2}. \tag{4.9}$$

To obtain $(4.9)_2$, write $(4.8)_2$ as

$$\nu = \frac{3\dfrac{K}{\mu} - 2}{2\left(3\dfrac{K}{\mu} + 1\right)} \tag{4.10}$$

and observe that $\nu \longrightarrow -1$ as $\dfrac{K}{\mu} \longrightarrow 0$, and $\nu \longrightarrow \dfrac{1}{2}$ as $\dfrac{K}{\mu} \longrightarrow \infty$.

The inequalities (4.7) imply that

(a) pressure produces a decrease in volume in dilatation ($K > 0$)
(b) the shear is in the same direction as the shear stress in simple shear ($\mu > 0$)
(c) axial tension results in axial elongation in simple extension ($E > 0$)

Thus inequalities (4.7) ensure physically reasonable response of the material in the three homogeneous deformations discussed above. However, $(4.9)_2$ permits the possibility of axial extension being accompanied by lateral expansion ($\nu < 0$). But no known isotropic material responds to simple extension in this way. Therefore in practice $0 < \nu < \dfrac{1}{2}$.

The strain-stress law. Deviatoric tensors

The stress-strain law (4.1) is repeated here for convenience:

$$\sigma_{ij} = \lambda e_{pp}\delta_{ij} + 2\mu e_{ij}.$$

Taking the trace of both sides gives

$$\sigma_{pp} = (3\lambda + 2\mu)e_{pp}. \tag{4.11}$$

Thus $\qquad \sigma_{ij} = \dfrac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\,\delta_{ij} + 2\mu e_{ij} \qquad$ and so

$$e_{ij} = \frac{1}{2\mu}\left\{-\frac{\lambda}{3\lambda + 2\mu}\sigma_{pp}\,\delta_{ij} + \sigma_{ij}\right\} \tag{4.12}$$

which is the **strain-stress** law, inverse to (4.1).

The **deviatoric stress** and **deviatoric strain** are defined by

$$\sigma'_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{pp}\,\delta_{ij}, \qquad e'_{ij} = e_{ij} - \frac{1}{3}e_{pp}\,\delta_{ij}, \tag{4.13}$$

respectively, and have the property that their traces vanish:

$$\sigma'_{pp} = 0, \qquad e'_{pp} = 0. \tag{4.14}$$

On substituting for σ_{ij} and e_{ij} from (4.13) into (4.1) we find that the stress-strain law (4.1) may be written

$$\sigma_{pp} = 3Ke_{pp}, \qquad \sigma'_{ij} = 2\mu e'_{ij}, \tag{4.15}$$

involving the bulk modulus K and the shear modulus μ.

From $(4.15)_1$ we may recover our previous definition (4.4) of bulk modulus:

$$K = \frac{\frac{1}{3}\sigma_{pp}}{e_{pp}}.$$

Equation $(4.15)_2$ makes clear the proportionality of shear stresses and strains, connected by the shear modulus μ.

Incompressibility

An **incompressible** material is one whose volume cannot be changed, though it can be changed in shape, i.e. distorted.

In an incompressible material, therefore, every deformation is such that the dilatation vanishes:

$$\Delta = e_{pp} = 0.$$

The uniform dilatation (4.3), i.e. $\mathbf{u} = \alpha\mathbf{x}$, cannot occur in an incompressible material as $e_{pp} = 0$ implies $\alpha = 0$.

For arbitrary hydrostatic pressure p there is no change of volume and so the stress-strain law (4.1) is replaced by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu e_{ij} \tag{4.16}$$

in which $p(\mathbf{x}, t)$ is an arbitrary pressure, not dependent on the strains e_{ij}.

The bulk modulus K is defined at (4.4) by

$$K = \frac{-p}{e_{pp}} = \lambda + \frac{2}{3}\mu.$$

In the limit of incompressibility $e_{pp} \longrightarrow 0$ and so

$$K \longrightarrow \infty, \qquad \lambda \longrightarrow \infty \tag{4.17}$$

and μ is unaltered.

For the simple shear discussed previously, $e_{pp} = 0$ and so simple shear is possible in any incompressible material.

For the uniaxial tension considered before:

$$u_1 = \alpha x_1, \qquad u_2 = -\beta x_2, \qquad u_3 = -\beta x_3$$

to be possible in an incompressible material the dilatation must vanish, so that

$$e_{pp} = \alpha - 2\beta = 0 \quad \Rightarrow \quad \frac{\beta}{\alpha} = \frac{1}{2}.$$

But Poisson's ratio is given by (4.6) to be

$$\nu = \frac{-e_{22}}{e_{11}} = \frac{\beta}{\alpha} = \frac{1}{2},$$

so that

$$\nu = \frac{1}{2} \tag{4.18}$$

for all incompressible materials. We can see this another way — from (4.6)

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

and for incompressibility $\lambda \longrightarrow \infty$, so that $\nu \longrightarrow \frac{1}{2}$, as at (4.18).

5 Conservation of energy: the strain energy function

The equations of motion (3.6) are

$$\frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}. \tag{5.1}$$

Multiply by the velocity $v_i = \partial u_i / \partial t$ and sum over i:

$$\frac{\partial \sigma_{ij}}{\partial x_j} v_i + \rho b_i v_i = \rho \frac{\partial v_i}{\partial t} v_i.$$

Thus

$$\frac{\partial}{\partial x_j}(\sigma_{ij} v_i) - \sigma_{ij} \frac{\partial v_i}{\partial x_j} + \rho b_i v_i = \frac{1}{2} \rho \frac{\partial}{\partial t}(v_i v_i). \tag{5.2}$$

Now

$$\frac{\partial v_i}{\partial x_j} = \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial u_i}{\partial x_j} = \frac{\partial e_{ij}}{\partial t} + \frac{\partial \omega_{ij}}{\partial t},$$

where the strain tensor $\mathbf{e}$ and rotation $\boldsymbol{\omega}$ are defined by

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \qquad \omega_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).$$

Then

$$\sigma_{ij} \frac{\partial v_i}{\partial x_j} = \sigma_{ij} \left(\frac{\partial e_{ij}}{\partial t} + \frac{\partial \omega_{ij}}{\partial t} \right),$$

But $\boldsymbol{\omega}$ is skew-symmetric, i.e. $\boldsymbol{\omega}^T = -\boldsymbol{\omega}$ or $\omega_{ji} = -\omega_{ij}$, so $\dfrac{\partial \boldsymbol{\omega}}{\partial t}$ also is skew-symmetric and it follows that $\sigma_{ij} \dfrac{\partial \omega_{ij}}{\partial t} = 0$.

Then (5.2) may be written

$$\frac{\partial}{\partial x_j}(\sigma_{ij} v_i) + \rho b_i v_i = \sigma_{ij} \frac{\partial e_{ij}}{\partial t} + \frac{1}{2} \rho \frac{\partial}{\partial t}(v^2).$$

Integrate over an arbitrary sub-region R_t of the body B_t and use the divergence theorem:

$$\int_{\partial R_t} \sigma_{ij} v_i n_j \, da + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} \, dv = \int_{R_t} \sigma_{ij} \frac{\partial e_{ij}}{\partial t} \, dv + \int_{R_t} \frac{1}{2} \rho \frac{\partial}{\partial t}(v^2) \, dv.$$

But $\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma} \mathbf{n}$ and ρ is assumed constant to this order:

$$\int_{\partial R_t} \mathbf{t}(\mathbf{n}) \cdot \mathbf{v} \, da + \int_{R_t} \rho \mathbf{b} \cdot \mathbf{v} \, dv = \int_{R_t} \sigma_{ij} \frac{\partial e_{ij}}{\partial t} \, dv + \frac{\partial}{\partial t} \int_{R_t} \frac{1}{2} \rho v^2 \, dv. \tag{5.3}$$

The terms of this equation are interpreted as follows:

1st term: rate of working of surface tractions
2nd term: rate of working of body forces
3rd term: stress power
4th term: rate of change of kinetic energy

The stress power per unit volume is defined by

$$P := \sigma_{ij}\frac{\partial e_{ij}}{\partial t}. \tag{5.4}$$

Suppose this is wholely derived as the rate of change of a single function $W(\mathbf{e})$, the **strain energy**, measured per unit volume, which depends on the strain $\mathbf{e}$:

$$\frac{\partial W}{\partial t} = \sigma_{ij}\frac{\partial e_{ij}}{\partial t}.$$

Now by the chain rule

$$\frac{\partial W}{\partial e_{ij}}\frac{\partial e_{ij}}{\partial t} = \sigma_{ij}\frac{\partial e_{ij}}{\partial t}$$

so that

$$\left(\frac{\partial W}{\partial e_{ij}} - \sigma_{ij}\right)\frac{\partial e_{ij}}{\partial t} = 0.$$

But at each position $\mathbf{x}$ and corresponding value of $\mathbf{e}$, $\dfrac{\partial e_{ij}}{\partial t}$ may be selected arbitrarily, so since the brackets do **not** depend on $\dfrac{\partial e_{ij}}{\partial t}$, we may conclude:

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}. \tag{5.5}$$

Thus $W(\mathbf{e})$ is a potential function for the stress $\boldsymbol{\sigma}$. It is often termed the **potential energy**. These ideas carry over into nonlinear elasticity.

The strain energy in isotropic linear elasticity

From (4.1) we see that the stress components σ_{ij} are a *linear* combination of the strain components e_{ij}. So from (5.5), $W(\mathbf{e})$ must be a homogeneous *quadratic* in the strain components.

From Euler's result on the partial differentiation of a homogeneous function of degree n:

[Namely, if $\phi(\lambda x_i) = \lambda^n\phi(x_i)$ then $\sum_{i=1}^{M} x_i\frac{\partial\phi}{\partial x_i} = n\phi$]

$$\frac{\partial W}{\partial e_{ij}}e_{ij} = 2W$$

so from (5.5)

$$W = \frac{1}{2}\sigma_{ij}e_{ij}. \tag{5.6}$$

Using the generalized Hooke's law (4.1) allows the strain energy to be expressed entirely in terms of the strain components e_{ij}:

$$W = \frac{1}{2}\lambda(e_{pp})^2 + \mu e_{ij}e_{ij}. \tag{5.7}$$

Written out in full

$$W = \frac{1}{2}\lambda(e_{11} + e_{22} + e_{33})^2 + \mu\{e_{11}^2 + e_{22}^2 + e_{33}^2 + 2e_{12}^2 + 2e_{23}^2 + 2e_{31}^2\}. \tag{5.8}$$

On physical grounds we expect the **quadratic form** (5.7) to be **positive definite**:

$$W(\mathbf{e}) \geq 0, \quad \text{with} \quad W(\mathbf{e}) = 0 \quad \text{only if} \quad \mathbf{e} = \mathbf{0}. \tag{5.9}$$

Any straining of the body ($\mathbf{e} \neq \mathbf{0}$) requires work to be done **on** the body ($W > 0$): $W < 0$ for some $\mathbf{e} \neq \mathbf{0}$ would imply that starting from an unstressed state of rest work could spontaneously be done **on** the surroundings.

We now seek necessary and sufficient conditions on λ and μ for $W(\mathbf{e})$ given by (5.8) to be positive definite. From (4.4) the bulk modulus is $K = \lambda + \frac{2}{3}\mu$ and elimination of λ from (5.8) in favour of K gives, after much manipulation,

$$W = \frac{1}{2}K(e_{11} + e_{22} + e_{33})^2 + \frac{1}{3}\mu\left\{(e_{11} - e_{22})^2 + (e_{22} - e_{33})^2 + (e_{33} - e_{11})^2\right\}$$
$$+ 2\mu\left\{e_{12}^2 + e_{23}^2 + e_{31}^2\right\}. \tag{5.10}$$

For the uniform dilatation $e_{11} = e_{22} = e_{33} = \alpha$, $e_{ij} = 0$ for $i \neq j$, (5.10) becomes

$$W = \frac{9K}{2}\alpha^2,$$

so that $W > 0$ for $\alpha \neq 0$ requires $K > 0$.
For the simple shear $e_{12} = e_{21} = \gamma/2$, all other e_{ij} vanishing, (5.10) becomes

$$W = \frac{\mu}{2}\gamma^2,$$

so that $W > 0$ for $\gamma \neq 0$ requires $\mu > 0$. Therefore, **necessary** conditions for the positive definiteness of W are

$$K > 0, \qquad \mu > 0. \tag{5.11}$$

These are also **sufficient** as they ensure that, from (5.10), W is expressed as a sum of squares with positive coefficients such that $W(\mathbf{e}) = 0$ requires $\mathbf{e} = \mathbf{0}$.

But conditions (5.11) are precisely those adopted before, on different physical grounds, see (4.7) and the following text.

At $(4.13)_2$ we introduced the strain deviator

$$e'_{ij} = e_{ij} - \frac{1}{3}e_{pp}\,\delta_{ij} \qquad e'_{pp} = 0, \tag{5.12}$$

in terms of which it may be shown that the strain energy W, taken in the form (5.7), may be written

$$W = \frac{1}{2}K(e_{pp})^2 + \mu e'_{ij}e'_{ij}. \tag{5.13}$$

Conditions (5.11) may also be deduced from this form of W.

6 The torsion problem

We consider an isotropic elastic cylinder of length l and arbitrary cross-section, placed with its generators parallel to the x_3-axis and with the origin 0 in one end face, as shown in Figure 12. The cylinder is twisted about $0x_3$ by an amount α per unit length by the application of shear tractions to its end faces, the curved surface remaining stress free. The total angle of twist, αl, is sufficiently small for its square to be neglected.

Figure 12: Cylinder under torsion

Figure 13: The cross-section S

Figure 13 gives a plan view of the typical right cross-section S indicated in Figure 12. If P is a typical point of S and $0'$ the point of intersection of $0x_3$ with S, the deformation rotates the line $0'P$ anticlockwise through an angle αx_3, as shown. The components of the displacement of P in the x_1- and x_2-directions are therefore, with $r = 0'P$, $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$,

$$\left.\begin{aligned} u_1 &= r\cos(\theta+\alpha x_3) - r\cos\theta &&= r\cos\theta\cos\alpha x_3 - r\sin\theta\sin\alpha x_3 - r\cos\theta \\ &= -r\sin\theta\,(\alpha x_3) + O(\alpha x_3)^2 &&= -\alpha x_2 x_3, \\ u_2 &= r\sin(\theta+\alpha x_3) - r\sin\theta &&= r\sin\theta\cos\alpha x_3 + r\cos\theta\sin\alpha x_3 - r\sin\theta \\ &= r\cos\theta\,(\alpha x_3) + O(\alpha x_3)^2 &&= \alpha x_1 x_3. \end{aligned}\right\} \quad (6.1)$$

It follows from (6.1) and (2.4) that $e_{11} = e_{22} = 0$. We assume that $e_{33} = 0$ also, the torsional deformation consisting entirely of shear with no axial extension. Thus $\partial u_3/\partial x_3 = 0$, implying that u_3 is a function of x_1 and x_2 only:

$$u_3 = \alpha\phi(x_1, x_2).$$

The function ϕ, specifying the axial deviation of S from its initially plane form, is called the **warping function**. The displacements of the simple torsion deformation are therefore

$$u_1 = -\alpha x_2 x_3, \qquad u_2 = \alpha x_1 x_3, \qquad u_3 = \alpha\phi(x_1, x_2), \tag{6.2}$$

with strain components

$$(e_{ij}) = \frac{\alpha}{2}\begin{pmatrix} 0 & 0 & \dfrac{\partial\phi}{\partial x_1} - x_2 \\ 0 & 0 & \dfrac{\partial\phi}{\partial x_2} + x_1 \\ \dfrac{\partial\phi}{\partial x_1} - x_2 & \dfrac{\partial\phi}{\partial x_2} + x_1 & 0 \end{pmatrix}$$

satisfying $e_{pp} = 0$, so that from (4.1) the stress components are

$$(\sigma_{ij}) = \alpha\mu\begin{pmatrix} 0 & 0 & \dfrac{\partial\phi}{\partial x_1} - x_2 \\ 0 & 0 & \dfrac{\partial\phi}{\partial x_2} + x_1 \\ \dfrac{\partial\phi}{\partial x_1} - x_2 & \dfrac{\partial\phi}{\partial x_2} + x_1 & 0 \end{pmatrix}. \tag{6.3}$$

The equilibrium equations (3.7) in the absence of body force are

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0. \tag{6.4}$$

The $i = 1$ and $i = 2$ equations are satisfied trivially and the $i = 3$ equation gives

$$\frac{\partial}{\partial x_1}\left(\frac{\partial\phi}{\partial x_1} - x_2\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial\phi}{\partial x_2} + x_1\right) = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} = \nabla^2\phi = 0, \tag{6.5}$$

the warping function thus being a **harmonic** function.

The outward unit normal $\mathbf{n}$ to C, the boundary of S, has components $(n_1, n_2, 0)$, see Figure 13. The components of the stress vector acting on the curved surface of the cylinder are, from (3.2) and (6.3),

$$\mathbf{t}(\mathbf{n}) = \boldsymbol{\sigma}\mathbf{n} \quad \text{or} \quad \begin{pmatrix} 0 & 0 & \sigma_{31} \\ 0 & 0 & \sigma_{32} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}\begin{pmatrix} n_1 \\ n_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma_{31}n_1 + \sigma_{32}n_2 \end{pmatrix}.$$

Since this surface is stress free,

$$\sigma_{31}n_1 + \sigma_{32}n_2 = 0 \quad \text{on} \quad C \tag{6.6}$$

and substituting for the stress components from (6.3) gives

$$\frac{\partial\phi}{\partial x_1}n_1 + \frac{\partial\phi}{\partial x_2}n_2 = x_2 n_1 - x_1 n_2 \quad \text{on} \quad C,$$

or

$$\frac{\partial\phi}{\partial n} = x_2 n_1 - x_1 n_2 \quad \text{on} \quad C, \tag{6.7}$$

$\partial\phi/\partial n$ being the directional derivative of ϕ along the outward normal to C. The warping function ϕ thus satisfies the Neumann problem consisting of the plane Laplace equation (6.5) in S together with the boundary condition (6.7) on C.

The twisting torque

The outward unit normal to the end face $x_3 = l$ has components $(0, 0, 1)$. The traction vector acting on this face therefore has components σ_{i3} and the components of the resultant force $\mathbf{F}$ acting on the face are,

$$F_i = \iint_S \sigma_{i3}\Big|_{x_3=l} \, da.$$

From (6.3), σ_{13} and σ_{23} are independent of x_3 and $\sigma_{33} = 0$ so

$$F_1 = \iint_S \sigma_{13} \, da, \qquad F_2 = \iint_S \sigma_{23} \, da, \qquad F_3 = 0.$$

From the $i = 3$ equilibrium equation (6.4) and the stress (6.3), and its symmetry

$$\frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} = \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0.$$

It follows that

$$\frac{\partial (x_1 \sigma_{13})}{\partial x_1} + \frac{\partial (x_1 \sigma_{23})}{\partial x_2} = x_1 \left(\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} \right) + \sigma_{13} = \sigma_{13},$$

$$\frac{\partial (x_2 \sigma_{13})}{\partial x_1} + \frac{\partial (x_2 \sigma_{23})}{\partial x_2} = x_2 \left(\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} \right) + \sigma_{23} = \sigma_{23},$$

and

$$F_1 = \iint_S \left\{ \frac{\partial (x_1 \sigma_{13})}{\partial x_1} + \frac{\partial (x_1 \sigma_{23})}{\partial x_2} \right\} da = \oint_C x_1 (\sigma_{31} n_1 + \sigma_{32} n_2) \, ds = 0,$$

$$F_2 = \iint_S \left\{ \frac{\partial (x_2 \sigma_{13})}{\partial x_1} + \frac{\partial (x_2 \sigma_{23})}{\partial x_2} \right\} da = \oint_C x_2 (\sigma_{31} n_1 + \sigma_{32} n_2) \, ds = 0,$$

using the two-dimensional divergence theorem and the boundary condition (6.6). Thus

$$\mathbf{F} = \mathbf{0},$$

and, in exactly the same way, the resultant force on the end face $x_3 = 0$ is zero.

The torque per unit area about 0 acting on the face $x_3 = l$ is

$$\mathbf{x} \times \mathbf{t}(\mathbf{e}_3) = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & l \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix},$$

and so, remembering that $\sigma_{33} = 0$, the resultant torque $\mathbf{M}$ acting on the face $x_3 = l$ has components

$$\begin{aligned} M_1 &= \textstyle\iint_S (x_2 \sigma_{33} - l \sigma_{32})\big|_{x_3=l} \, da &= -lF_2 &= 0, \\ M_2 &= \textstyle\iint_S (l \sigma_{31} - x_1 \sigma_{33})\big|_{x_3=l} \, da &= lF_1 &= 0, \\ M_3 &= \textstyle\iint_S (x_1 \sigma_{32} - x_2 \sigma_{31})\big|_{x_3=l} \, da && \\ &= \mu\alpha \iint_S \left\{ x_1 \left(\frac{\partial \phi}{\partial x_2} + x_1 \right) - x_2 \left(\frac{\partial \phi}{\partial x_1} - x_2 \right) \right\} da &= \alpha D, && \end{aligned}$$

where

$$D = \mu \iint_S \left(x_1^2 + x_2^2 + x_1 \frac{\partial \phi}{\partial x_2} - x_2 \frac{\partial \phi}{\partial x_1} \right) da. \tag{6.8}$$

Thus $\mathbf{M} = \alpha D \mathbf{e}_3$, where $\mathbf{e}_3$ is the unit vector in the direction of x_3 increasing, and the resultant torque on the end face $x_3 = 0$ is similarly $-\alpha D \mathbf{e}_3$.

The cylinder is therefore maintained in equilibrium by equal and opposite twisting torques of magnitude αD acting about $0x_3$ on the end faces. The quantity D, defined by (6.8), is called the **torsional rigidity** of S. D/μ has physical dimension $(\text{length})^4$ and depends only on the form of the cross-section S.

The Prandtl stress function

Since ϕ is a plane harmonic function, there exists an analytic function f of the complex variable $z = x_1 + ix_2$ such that

$$\phi(x_1, x_2) = \Re\text{e}\, f(z).$$

Let

$$\psi(x_1, x_2) = \Im\text{m}\, f(z),$$

so that

$$f(x_1 + ix_2) = \phi(x_1, x_2) + i\psi(x_1, x_2).$$

Then ϕ and ψ are conjugate harmonic functions satisfying the Cauchy-Riemann equations

$$\frac{\partial \phi}{\partial x_1} = \frac{\partial \psi}{\partial x_2}, \qquad \frac{\partial \phi}{\partial x_2} = -\frac{\partial \psi}{\partial x_1}. \tag{6.9}$$

The unit vector $\mathbf{t}$ tangential to C in the anticlockwise sense has components

$$t_1 = -n_2, \qquad t_2 = n_1, \qquad t_3 = 0, \tag{6.10}$$

see Figure 13. Using (6.9) and (6.10) we can rewrite (6.7) as follows:

$$\begin{aligned} \frac{\partial \phi}{\partial x_1} n_1 + \frac{\partial \phi}{\partial x_2} n_2 &= x_2 n_1 - x_1 n_2 \quad \text{on} \quad C, \\ \frac{\partial \psi}{\partial x_2} t_2 + \left(-\frac{\partial \psi}{\partial x_1} \right) (-t_1) &= x_2 t_2 - x_1 (-t_1) \quad \text{on} \quad C, \\ \frac{\partial \psi}{\partial x_1} t_1 + \frac{\partial \psi}{\partial x_2} t_2 &= x_1 t_1 + x_2 t_2 \quad \text{on} \quad C. \end{aligned}$$

Now by definition

$$\frac{\partial \psi}{\partial x_1} t_1 + \frac{\partial \psi}{\partial x_2} t_2 = \mathbf{t} \cdot \nabla \psi = \frac{\partial \psi}{\partial s},$$

where s is arc length along C in the positive (anti-clockwise) sense. So (6.7) finally reduces to

$$\frac{\partial \psi}{\partial s} = \frac{1}{2} \frac{\partial}{\partial s} (x_1^2 + x_2^2) \quad \text{on} \quad C.$$

The Neumann problem (6.5), (6.7) is thus equivalent to the Dirichlet problem

$$\left.\begin{aligned} \nabla^2 \psi &= 0 \quad \text{in} \quad S, \\ \psi &= \tfrac{1}{2}(x_1^2 + x_2^2) + \text{constant on } C. \end{aligned}\right\} \tag{6.11}$$

Simpler results are obtained by introducing the **Prandtl stress function**

$$\Psi = \psi - \frac{1}{2}(x_1^2 + x_2^2). \tag{6.12}$$

From (6.11), Ψ satisfies the Poisson equation

$$\nabla^2\Psi = -2 \quad \text{in} \quad S, \tag{6.13}$$

and the boundary condition

$$\Psi = \Psi_0 \quad \text{on} \quad C, \tag{6.14}$$

where Ψ_0 is a constant.

In terms of Ψ, the non-zero shear stresses are given from (6.3), (6.9) and (6.12) by

$$\left.\begin{aligned} \sigma_{23} &= \mu\alpha\left(\frac{\partial\phi}{\partial x_2} + x_1\right) = -\mu\alpha\left(\frac{\partial\psi}{\partial x_1} - x_1\right) = -\mu\alpha\frac{\partial\Psi}{\partial x_1} \\ \sigma_{31} &= \mu\alpha\left(\frac{\partial\phi}{\partial x_1} - x_2\right) = \mu\alpha\left(\frac{\partial\psi}{\partial x_2} - x_2\right) = \mu\alpha\frac{\partial\Psi}{\partial x_2}, \end{aligned}\right\} \tag{6.15}$$

and the torsional rigidity from (6.8), (6.9) and (6.12) by

$$D = \mu\iint_S \left(x_1^2 + x_2^2 - x_1\frac{\partial\psi}{\partial x_1} - x_2\frac{\partial\psi}{\partial x_2}\right) da = -\mu\iint_S \left(x_1\frac{\partial\Psi}{\partial x_1} + x_2\frac{\partial\Psi}{\partial x_2}\right) da. \tag{6.16}$$

Since

$$x_1\frac{\partial\Psi}{\partial x_1} + x_2\frac{\partial\Psi}{\partial x_2} = \frac{\partial(x_1\Psi)}{\partial x_1} + \frac{\partial(x_2\Psi)}{\partial x_2} - 2\Psi,$$

the application of the two-dimensional divergence theorem gives

$$\begin{aligned} \iint_S \left(x_1\frac{\partial\Psi}{\partial x_1} + x_2\frac{\partial\Psi}{\partial x_2}\right) da &= -2\iint_S \Psi\, da + \mu\oint_C \Psi(x_1n_1 + x_2n_2)ds \\ &= -2\iint_S \Psi\, da + \mu\oint_C \Psi_0(x_1n_1 + x_2n_2)ds, \end{aligned}$$

where (6.14) has been used. Thus

$$D = 2\iint_S \Psi\, da - \mu\oint_C \Psi_0(x_1n_1 + x_2n_2)ds. \tag{6.17}$$

If S is simply connected, so that C is a **single** closed curve we can replace Ψ by $\Psi - \Psi_0$. Equations (6.13), (6.15) and (6.16) are unchanged and (6.14) and (6.17) simplify to

$$\Psi = 0 \text{ on } C, \tag{6.18}$$

$$D = 2\mu\iint_S \Psi\, da. \tag{6.19}$$

Figure 14: The elliptical cylinder

Examples

1. Torsion of an elliptical cylinder

Suppose that C is an ellipse and that $0x_3$ is the line of centres of the right cross-sections. If a_1 and a_2 are the semi-axes of the ellipse in the x_1- and x_2-directions, the equation of C is

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1.$$

The function

$$\Psi(x_1, x_2) = A\left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - 1\right)$$

clearly satisfies the boundary condition (6.18), i.e. vanishes on C, and

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x_1^2} + \frac{\partial^2\Psi}{\partial x_2^2} = A\left(\frac{2}{a_1^2} + \frac{2}{a_2^2}\right).$$

The Poisson equation (6.13) therefore holds if

$$A\left(\frac{2}{a_1^2} + \frac{2}{a_2^2}\right) = -2 \quad \Rightarrow \quad A = -\frac{a_1^2 a_2^2}{a_1^2 + a_2^2},$$

and so the Prandtl stress function for the elliptical cylinder is

$$\Psi = -\frac{a_1^2 a_2^2}{a_1^2 + a_2^2}\left(\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} - 1\right) = \frac{a_1^2 a_2^2 - a_2^2 x_1^2 - a_1^2 x_2^2}{a_1^2 + a_2^2}.$$

From (6.12), i.e. $\psi = \Psi + \frac{1}{2}(x_1^2 + x_2^2)$,

$$\begin{aligned}
\psi(x_1, x_2) &= \frac{a_1^2 a_2^2 - a_2^2 x_1^2 - a_1^2 x_2^2}{a_1^2 + a_2^2} + \frac{1}{2}(x_1^2 + x_2^2) \\
&= \frac{1}{2(a_1^2 + a_2^2)}(2a_1^2 a_2^2 - 2a_2^2 x_1^2 - 2a_1^2 x_2^2 + a_1^2 x_1^2 + a_1^2 x_2^2 + a_2^2 x_1^2 + a_2^2 x_2^2) \\
&= \frac{1}{2(a_1^2 + a_2^2)}\left\{(a_1^2 - a_2^2)(x_1^2 - x_2^2) + 2a_1^2 a_2^2\right\} \\
&= \Im\mathrm{m}\,\frac{1}{2(a_1^2 + a_2^2)}\left\{(a_1^2 - a_2^2)iz^2 + 2ia_1^2 a_2^2\right\}
\end{aligned}$$

where $z = x_1 + ix_2$ and so $z^2 = x_1^2 - x_2^2 + 2ix_1x_2$ and $iz^2 = i(x_1^2 - x_2^2) - 2x_1x_2$. The **warping function** ϕ is therefore

$$\begin{aligned}\phi(x_1, x_2) &= \Re\mathrm{e}\,\frac{1}{2(a_1^2 + a_2^2)}\left\{(a_1^2 - a_2^2)iz^2 + 2ia_1^2a_2^2\right\} \\ &= -\frac{a_1^2 - a_2^2}{a_1^2 + a_2^2}\,x_1x_2.\end{aligned}$$

When $a_1 > a_2$ the warping pattern is as shown in Figure 15. The curves $\phi =$ constant are rectangular hyperbolae.

Figure 15: The warping pattern for an elliptical cylinder

The **torsional rigidity** D is, from (6.19),

$$D = 2\mu \iint_S \Psi\, da = \frac{2\mu a_1^2 a_2^2}{a_1^2 + a_2^2}\iint_S \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right) dx_1 dx_2.$$

Calculation of D. Because S is elliptical we transform the double integral using modified polar coordinates

$$x_1 = a_1 r\cos\theta, \qquad x_2 = a_2 r\sin\theta.$$

Since $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = r^2$ the limits of integration are $0 \le r \le 1,\ 0 \le \theta \le 2\pi$.

The Jacobian is

$$\frac{\partial(x_1, x_2)}{\partial(r, \theta)} = \begin{vmatrix} \dfrac{\partial x_1}{\partial r} & \dfrac{\partial x_1}{\partial \theta} \\ \dfrac{\partial x_2}{\partial r} & \dfrac{\partial x_2}{\partial \theta}\end{vmatrix} = \begin{vmatrix} a_1\cos\theta & -a_1 r\sin\theta \\ a_2\sin\theta & a_2 r\cos\theta \end{vmatrix}$$

$$= a_1a_2r(\cos^2\theta + \sin^2\theta) = a_1a_2r,$$

so that $dx_1dx_2 \ \mapsto\ a_1a_2\, r dr d\theta$. Then the double integral is

$$\iint_S \left(1 - \frac{x_1^2}{a_1^2} - \frac{x_2^2}{a_2^2}\right) dx_1dx_2 = \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(1 - \frac{a_1^2r^2\cos^2\theta}{a_1^2} - \frac{a_2^2r^2\sin^2\theta}{a_2^2}\right) a_1a_2\, r dr d\theta$$

$$= a_1 a_2 \int_{\theta=0}^{2\pi}\int_{r=0}^{1}\left(1-r^2\right) r dr d\theta = 2\pi a_1 a_2 \left[\frac{r^2}{2}-\frac{r^4}{4}\right]_0^1 = \frac{\pi a_1 a_2}{2}.$$

Therefore

$$D = \frac{2\mu a_1^2 a_2^2}{a_1^2+a_2^2} \times \frac{\pi a_1 a_2}{2} = \frac{\pi \mu a_1^3 a_2^3}{a_1^2+a_2^2},$$

the torsional rigidity of an elliptical cylinder.

Torsion of a circular cylinder In the case of a circular cylinder, $a_1 = a_2 = a$ and

$$\begin{aligned} \Psi(x_1, x_2) &= \frac{1}{2}(a^2 - x_1^2 - x_2^2), \\ \phi(x_1, x_2) &= 0, \\ D &= \frac{1}{2}\pi\mu a^4. \end{aligned}$$

Because $\phi = 0$, there is no warping when the circular cylinder is twisted about its central axis.

2. Torsion of a circular tube

Figure 16: The circular tube

We consider the cross-section S bounded by the concentric circles C_1, C_2 of radii a and b ($< a$). The axis of torsion is once again the axis of symmetry.

The function

$$\Psi(x_1, x_2) = \frac{1}{2}(a^2 - x_1^2 - x_2^2),$$

obtained above for the circular cylinder, satisfies the Poisson equation (6.13), vanishes on C_1 and is constant on C_2. It is therefore an acceptable Prandtl stress function for the circular tube and, as before, $\phi = 0$, so that the right cross-sections are unwarped. Because the boundary of the tube consists of disjoint curves we cannot use the fomula (6.19) for the torsional rigidity. Instead, we obtain from (6.16) the expression

$$\begin{aligned} D &= -\mu \iint_S \left(-x_1^2 - x_2^2\right) da = \mu \int_b^a r^2 . 2\pi r dr = 2\pi\mu \left[\frac{1}{4} r^4\right]_b^a \\ &= \frac{1}{2}\pi\mu(a^4 - b^4) \end{aligned}$$

for the torsional rigidity of a circular tube.