“Forcing Idealized” by J. Zapletal, a review

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At the time I was a young postdoc, so about the same time when Jindřa Zapletal was finishing his Ph.D., the accepted wisdom of the junior community of set theorists working in the set theory of the reals was that large cardinals were a fancy useless invention one better steer clear off. The accepted wisdom of the ones working in large cardinals was that the set theory of the reals consisted of infinitely many intellectually isomorphic ways to juggle infinitely many cardinal invariants between the values of $\aleph_1$ and $\aleph_2$. Zapletal's work over the years, and culminating in this book, proved them all wrong by discovering a deep connection between these two parts of set theory.

To start with, this is a book about forcing of the form P_I, consisting of the positive Borel sets of a σ-ideal I on a Polish space, ordered by inclusion. Such a forcing adds a single generic point which is exactly the intersection of all sets in the generic filter (Proposition 2.1.2). Examples are easy to think of- say random reals- but it is rather surprising to realise that in the presence of suitable large cardinals, a very large class of forcings is of this kind (e.g. Proposition 2.1.8). A basic concept here is that under large cardinal assumptions ‘somewhat definable’ ideals behave as if they were indeed definable, so what is known about Borel σ-ideals can be extrapolated to larger classes of ideals. How large the class depends on how large the cardinal. An interesting class of forcing that emerges through this type of consideration are universally Baire forcings and in this class large cardinal assumptions give striking dichotomy results and characterisations. For example if we assume LC=large cardinals *and* CH, then every universally Baire forcing is either proper or collapses $\aleph_1$ below some condition. The previous theorem is connected with the old question of how close is the notion of properness to saying that the forcing preserves $\aleph_1$, or stationary subsets of ω_1. It is known that these notions are not equivalent. However, Zapletal's result which says that under LC a universally Baire forcing P is proper iff no condition makes

$([P]^{\aleph_0})^V$ nonstationary basically says that morally speaking properness is equivalent to preserving stationary sets, if we are happy to remain within the class of forcings which are reasonably definable and regular.

To quote from the book 'a careful review of this book will reveal that it is full of dichotomies and the dichotomies are really the driving force behind most arguments'. The results quoted above are dichotomies, and they are obtained by using more general dichotomies which under the large cardinal assumptions apply to universally Baire forcing. The choice of the name 'universally Baire' is not a coincidence, there is a strong connection with the known regularity properties of universally Baire sets of reals under large cardinal assumptions, although the connection is a little too technical to go into here. Universally Baire forcing is defined in Definition 2.1.7, and the basic idea is that the forcing conditions and the generic code a universally Baire set.

The book considers a myriad of properties 'idealized forcing' might have, adding this kind of real or that kind of real, how to deal with cardinal invariants, where do the classical forcings fit in, everything that a set theorist of the reals might ask. There is then naturally a lot about iterations, preservation theorems, even ill-founded iterations, games, applications. The book starts by stating that it is not a textbook. True. It is hard to read, for various reasons, the depth of the subject not being the least important one. The book is an excellent reference book and an excellent research monograph as well. Those who persevere will be rewarded not just with some good maths but with some good fun too: "What pentagram is to heavy metal, Cichoń's diagram is to set theory".

The research presented in this book has already had an excellent audience, including some young researchers working in the field, most promisingly a recent Wrocław graduate Marcin Sabok whose work has continued the themes investigated in the book.

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