

On Ordinal Invariants in Well Quasi Orders and Finite Antichain Orders^{*}

Mirna Džamonja, Sylvain Schmitz, and Philippe Schnoebelen

Abstract We investigate the ordinal invariants height, length, and width of well quasi orders (WQO), with particular emphasis on width, an invariant of interest for the larger class of orders with finite antichain condition (FAC). We show that the width in the class of FAC orders is completely determined by the width in the class of WQOs, in the sense that if we know how to calculate the width of any WQO then we have a procedure to calculate the width of any given FAC order. We show how the width of WQO orders obtained via some classical constructions can sometimes be computed in a compositional way. In particular this allows proving that every ordinal can be obtained as the width of some WQO poset. One of the difficult questions is to give a complete formula for the width of Cartesian products of WQOs. Even the width of the product of two ordinals is only known through a complex recursive formula. Although we have not given a complete answer to this question we have advanced the state of knowledge by considering some more complex special cases and in particular by calculating the width of certain products containing three factors. In the course of writing the paper we have discovered that some of the relevant literature was written on cross-purposes and some of the notions re-discovered

M. Džamonja,
University of East Anglia, Norwich, UK, e-mail: m.dzamonja@uea.ac.uk

S. Schmitz,
LSV, ENS Paris-Saclay & CNRS, Université Paris-Saclay, France, e-mail: schmitz@lsv.fr

Ph. Schnoebelen
LSV, CNRS & ENS Paris-Saclay, Université Paris-Saclay, France, e-mail: phs@lsv.fr

^{*} Mirna Džamonja thanks the Leverhulme Trust for a Research Fellowship for the academic year 2014/2015, the Simons Foundation for a Visiting Fellowship in the Autumn of 2015 and the Isaac Newton Institute in Cambridge for their support during the HIF programme in the Autumn of 2015, supported by EPSRC Grant Number EP/K032208/1. She most gratefully acknowledges the support of the *Institut d'histoire et de philosophie des sciences et des techniques (IHPST)* at University Paris 1, where she is as an Associate Member. The three authors thank the London Mathematical Society for their support through a Scheme 3 Grant.

several times. Therefore we also use the occasion to give a unified presentation of the known results.

Key words: wqo, width of wqo, ordinal invariants

1 Introduction

In the finite case, a partial order—also called a *poset*— (P, \leq) has natural cardinal invariants: a *width*, which is the cardinal of its maximal antichains, and a *height*, which is the cardinal of its maximal chains. The width and height are notably the subject of the theorems of Dilworth (1950) and Mirsky (1971) respectively; see West (1982) for a survey of these *extremal* problems. In the infinite case, cardinal invariants are however less informative—especially for countable posets—, while the theorems of Dilworth and Mirsky are well-known to fail (Peles, 1963; Schmerl, 2002).

When the poset at hand enjoys additional conditions, the corresponding *ordinal invariants* offer a richer theory, as studied for instance by Kříž and Thomas (1990). Namely, if (P, \leq) has the *the finite antichain condition* (FAC), meaning that its antichains are finite, then the tree

$$\text{Inc}(P) \stackrel{\text{def}}{=} \{ \langle x_0, x_1, \dots, x_n \rangle \in P^{<\omega} : 0 \leq n < \omega \wedge \forall 0 \leq i < j \leq n, x_i \perp x_j \}$$

of all non-empty (finite) sequences of pairwise *incomparable* elements of P ordered by initial segments has no infinite branches. Note that the tree $(\text{Inc}(P), \triangleleft)$ does not necessarily have a single root and that the empty sequence is excluded (the latter is a matter of aesthetics, but it does make various arguments run more smoothly by not having to consider the case of the empty sequence separately). Therefore, $\text{Inc}(P)$ has a rank, which is the smallest ordinal γ such that there is a function $f : \text{Inc}(P) \rightarrow \gamma$ with $s \triangleleft t \implies f(s) > f(t)$ for all $s, t \in \text{Inc}(P)$. This ordinal is called the *width* of P and in this paper we denote it by $\mathbf{w}(P)$ —it was denoted by $\text{wd}(P)$ by Kříž and Thomas (1990).

Similarly, if (P, \leq) is *well-founded* (WF), also called Artinian, meaning that its descending sequences are finite, then the tree

$$\text{Dec}(P) \stackrel{\text{def}}{=} \{ \langle x_0, x_1, \dots, x_n \rangle \in P^{<\omega} : 0 \leq n < \omega \wedge \forall 0 \leq i < j \leq n, x_i > x_j \}$$

of non-empty strictly descending sequences has an ordinal rank, which we denote by $\mathbf{h}(P)$ (Kříž and Thomas denote it by $\text{ht}(P)$) and call the *height* of P .

Finally, if (P, \leq) is both well-founded and FAC, i.e., is a *well partial order* (WPO), then the tree

$$\text{Bad}(P) \stackrel{\text{def}}{=} \{ \langle x_0, x_1, \dots, x_n \rangle \in P^{<\omega} : 0 \leq n < \omega \wedge \forall 0 \leq i < j \leq n, x_i \not\leq x_j \}$$

of non-empty *bad sequences* of P has an ordinal rank, which we denote by $\mathfrak{o}(P)$ and call the *maximal order type* of P after de Jongh and Parikh (1977) and Schmidt (1979) (Kříž and Thomas denote it by $c(P)$), Blass and Gurevich call it the *stature* of P). In the finite case, this invariant is simply the cardinal of the poset.

Quite some work has already been devoted to heights and maximal order types, and to their computation. Widths are however not that well-understood: as Kříž and Thomas (1990, Rem. 4.14) point out, they do not enjoy nice characterisations like heights and maximal order types do, and the range of available results and techniques on width computations is currently very limited.

Our purpose in this paper is to explore to what extent we can find such a characterisation, and provide formulæ for the behaviour of the width function under various classically defined operations with partial orders. Regarding the first point, we first show in Sec. 3 that the width coincides with the *antichain rank* defined by Abraham and Bonnet (1999), which is the height of the chains of antichains; however, unlike the height and maximal order type of WPOs, the width might not be attained (Rem. 3.7). Regarding the second point, we first show in Sec. 2.6 that computing widths in the class of FAC orders reduces to computing widths in the class of WPOs. We recall several techniques for computing ordinal invariants, and apply them in Sec. 4 to obtain closed formulæ for the width of sums of posets, and for the finite multisets, finite sequences, and tree extensions of WPOs. One of the main questions is to give a complete formula for the width of the Cartesian products of WPOs. Even the width of the product of two ordinals is only known through a complex recursive formula (due to Abraham, see Sec. 4.4) and we only have partial answers to the general question.

The three ordinal invariants appear in different streams of the literature, often unaware of the results appearing in one another, and using different definitions and notations. Another motivation of this paper is then to provide a unified presentation of the state of the knowledge on the subject, and we also recall the corresponding results for heights and maximal order types as we progress through the paper.

2 Background and Basic Results

2.1 Posets and Quasi-Orders

We consider posets and, more generally, quasi-orders (QO). When (Q, \leq_Q) is a QO, we write $x <_Q y$ when $x \leq_Q y$ and $y \not\leq_Q x$. We write $x \perp_Q y$ when $x \not\leq_Q y$ and $y \not\leq_Q x$, and say that a and b are *incomparable*. We write $x \equiv_Q y$ when $x \leq_Q y \wedge y \leq_Q x$: this is an equivalence and the quotient $(Q, \leq_Q) / \equiv_Q$ is a poset that, as far as ordinal invariants are concerned, is indistinguishable from Q . Therefore we restrict our attention to posets for technical reasons but without any loss of generality. Note that some constructions on posets (e.g., taking powersets) yield quasi-orders that are

not posets. A QO Q is *total* if for all x, y in Q , $x \leq_Q y$ or $x \geq_Q y$; a total poset is also called a *chain*.

When a QO does not have infinite antichains, we say that it satisfies the *Finite Antichain Condition*, or simply that it is FAC. A QO that does not have any infinite (strictly) decreasing sequence is said to be well-founded (or WF). A *well-quasi order* (or WQO) is a QO that is both WF and FAC: it is well-known that a QO is WQO if and only if it does not have any infinite bad sequence Kruskal (1972); Milner (1985), where a sequence $\langle x_0, x_1, x_2, \dots \rangle$ is *good* if $x_i \leq x_j$ for some positions $i < j$, and is *bad* otherwise.

For a QO (Q, \leq) we define the *reverse* QO Q^* as (Q, \geq) , that is to say, $x \leq_{Q^*} y$ if and only if $x \geq_Q y$. An *augmentation* of (Q, \leq) is a QO (Q, \le') such that $x \leq y \implies x \le' y$, i.e., \leq is a subset of \le' . A *substructure* of a QO (Q, \leq) is a QO (Q', \le') such that $Q' \subseteq Q$ and $\le' \subseteq \leq$. In this case, we write $Q' \leq Q$.

2.2 Rankings and Well-Founded Trees

Recall that for every WF poset P there exist ordinals γ and order preserving functions $f: P \rightarrow \gamma$, that is, such that $x <_P y \implies f(x) < f(y)$ for all $x, y \in P$. The smallest such ordinal γ is called the *rank* of P ; one can obtain the associated *ranking function* $r: P \rightarrow \gamma$ by defining inductively $r(x) = \sup\{r(y) + 1 : y <_P x\}$, and the rank turns out to be equal to its height $\mathbf{h}(P)$ (see Sec. 2.3). When P is total, i.e., is a chain, then its rank is also called its *order type*.

Traditionally, for a tree (T, \leq_T) , one says that it is well-founded if it *does not have an infinite branch*, which with the notation above amounts to saying that the reverse partial order (T, \geq_T) is well-founded. This somewhat confusing notation, implies that for rooted well-founded trees, the root(s) have the largest rank, and the leaves have rank 0. In our definitions of ordinal invariants given in the introduction, we considered trees of non-empty finite sequences, ordered by initial segments: if $s = \langle x_0, x_1, \dots, x_n \rangle$ and $t = \langle y_0, y_1, \dots, y_m \rangle$, we write $s \trianglelefteq t$ and say that s is an *initial segment* of t , when $n \leq m$ and $s = \langle y_0, \dots, y_n \rangle$. Equivalently, the associated strict ordering $s \triangleleft t$ means that t can be obtained by appending some sequence t' after s , denoted $t = s \frown t'$.

We also make an easy but important observation regarding substructures: When P is embedded in Q as an induced substructure, then $\mathbf{w}(P) \leq \mathbf{w}(Q)$, and similarly for \mathbf{o} and \mathbf{h} . Indeed, every antichain (bad sequence, decreasing sequence, resp.) of P is an antichain (bad sequence, decreasing sequence, resp.) of Q , so the ranks of the corresponding trees can only increase when going from P to Q .

2.3 Residual Characterisation

For a poset (P, \leq) , $x \in P$, and $* \in \{\perp, <, \not\leq\}$, we define the $*$ -residual of P at x as the induced poset defined by

$$P_{*x} \stackrel{\text{def}}{=} \{y \in P : y * x\}. \quad (1)$$

Since this is an induced substructure of P , P_{*x} is FAC (resp. WF, WPO) whenever P is FAC (resp. WF, WPO).

The interest of \perp -residuals (resp. $<$ -residuals, $\not\leq$ -residuals) is that they provide the range of choices for continuing incomparable (resp. descending, bad) sequences once element x has been chosen as first element: the suffix of the sequence should belong to P_{*x} , and we have recursively reduced the problem to measuring the rank of the tree $\text{Inc}(P_{\perp x})$ (resp. $\text{Dec}(P_{< x})$, $\text{Bad}(P_{\not\leq x})$).

The following lemma shows precisely how we can extract the rank from such a recursive decomposition of the tree.

Lemma 2.1.

1. Suppose that $\{T_i : i \in I\}$ is a family of well-founded trees and let T be their disjoint union. Then T is a well-founded tree and it has rank $\rho(T) = \sup_{i \in I} \rho(T_i)$.
2. Let $T = t \frown F$ denote a tree rooted at t with $F = T \setminus t$ and suppose that F is well-founded of rank $\rho(F)$. Then so is T , and $\rho(T) = \rho(F) + 1$.

Proof (of 1). It is clear that T is well founded. For each $i \in I$, let $f_i: T_i \rightarrow \rho(T_i)$ be a function witnessing the rank of T_i . Then $f \stackrel{\text{def}}{=} \bigcup_{i \in I} f_i$ is an order reversing function from T to $\gamma \stackrel{\text{def}}{=} \sup_{i \in I} \rho(T_i)$, showing $\rho(T) \leq \gamma$.

Conversely, if $f: T \rightarrow \rho(T)$ is a witness function for the rank of T , its restriction to any T_i is order reversing, showing that $\rho(T_i) \leq \rho(T)$.

Proof (of 2). Clearly T is well-founded. Let $\rho^* \stackrel{\text{def}}{=} \rho(F) + 1 = (\sup_{\alpha < \rho(F)} (\alpha + 1)) + 1$. Consider the ranking function $r: F \rightarrow \rho(F)$, and let $f: T \rightarrow \rho^*$ be given by

$$f(s) \stackrel{\text{def}}{=} \begin{cases} r(s) & \text{if } s \in F, \\ \sup_{\alpha < \rho(F)} (\alpha + 1) & \text{if } s = t. \end{cases}$$

It is clear that f is an order reversing function, witnessing $h(T) \leq \rho^*$. Suppose that $\beta < \rho^*$ and that $h: T \rightarrow \beta$ is an order reversing function. In particular, $h(r) < f(r)$, so let $\alpha < \rho(F)$ be such that $h(r) < \alpha + 1$. Let $s \in F$ be such that $f(s) = \alpha$. Hence $h(r) \leq h(s)$, yet $r <_T s$, a contradiction. ★2.1

Lemma 2.1 yields the equations:

$$\mathbf{w}(P) = \sup_{x \in P} \{\mathbf{w}(P_{\perp x}) + 1\}, \quad \mathbf{h}(P) = \sup_{x \in P} \{\mathbf{h}(P_{< x}) + 1\}, \quad \mathbf{o}(P) = \sup_{x \in P} \{\mathbf{o}(P_{\not\leq x}) + 1\}, \quad (2)$$

that hold for any FAC, WF, or WPO, poset P respectively. Note that it yields $\mathbf{w}(\emptyset) = \mathbf{h}(\emptyset) = \mathbf{o}(\emptyset) = 0$.

Equation (2) is used very frequently in the literature and provides for a method for computing ordinal invariants recursively, which we call the *method of residuals*.

Equation 2 further shows that the function $r(x) \stackrel{\text{def}}{=} \mathbf{h}(P_{<x})$ is the optimal ranking function of P . Thus $\mathbf{h}(P)$ is the rank of P , i.e. the minimal γ such that there exists a strict order-preserving $f: P \rightarrow \gamma$ (recall Sec. 2.2).

2.4 Games for WQO Invariants

One limitation of the method of residuals is that it tends to produce recursive rather than closed formulæ, see, e.g., Schmitz and Schnoebelen (2011). Another proof technique adopts a game-theoretical point of view. This is based on (Blass and Gurevich, 2008, §3), which in turn can be seen as an application of a classical game for the rank of trees to the specific trees used for the ordinal invariants. We shall use this technique to obtain results about special products of more than two orders, see for example Thm. 4.18.

The general setting is as follows. For a WQO P and an ordinal α , the game $G_{P,\alpha}^*$ —where $*$ is one of $\mathbf{h}, \mathbf{o}, \mathbf{w}$ —is a two-player game where positions are pairs (β, S) of an ordinal and a sequence over P . We start in the initial position $(\alpha, \langle \rangle)$. At each turn, and in position (β, S) , Player 1 picks an ordinal $\beta' < \beta$ and Player 2 answers by extending S with an element x from P . Player 2 is only allowed to pick x so that the extended $S' = S \frown x$ is a decreasing sequence (or a bad sequence, or an antichain) when $*$ = \mathbf{h} (resp. $*$ = \mathbf{o} , or $*$ = \mathbf{w}) and he loses the game if he cannot answer Player 1's move. After Player 2's move, the new position is (β', S') and the game continues. Player 2 wins when the position has $\beta = 0$ and hence Player 1 has no possible move. The game cannot run forever so one player has a winning strategy. Applying (Blass and Gurevich, 2008, Prop. 23) we deduce that Player 2 wins in $G_{P,\alpha}^*$ iff $*(P) \geq \alpha$. As we are mostly interested in the invariant \mathbf{w} , we shall adopt the notation $G_{P,\alpha}$ for $G_{P,\alpha}^{\mathbf{w}}$.

2.5 Cardinal Invariants

We can connect the ordinal invariants with cardinal measures but this does not lead to very fine bounds. Here are two examples of what can be said.

Lemma 2.2. *Suppose that Q is a FAC quasi-order of cardinal $\kappa \geq \aleph_0$. Then $\mathbf{w}(Q) < \kappa^+$, the cardinal successor of $|Q|$.*

Proof. The tree $\text{Inc}(Q)$ has size equal to κ and therefore its rank is an ordinal $\gamma < \kappa^+$. ★2.2

Theorem 2.3 (Dushnik-Miller). *Suppose that P is a WPO of cardinal $\kappa \geq \aleph_0$. Then $h(P) \geq \kappa$.*

Proof. This is an easy consequence of Thm. 5.25 in Dushnik and Miller (1941). By the definition of h , it suffices to show that P has a chain of size κ . Define a colouring c on the set $[P]^2$ of pairs of P by saying $c(x, y) \stackrel{\text{def}}{=} 0$ if x is comparable to y and $c(x, y) \stackrel{\text{def}}{=} 1$ otherwise. Then use the relation $\kappa \longrightarrow (\kappa, \aleph_0)^2$, meaning that P has a chain of cardinal κ or an antichain of cardinal \aleph_0 , which for $\kappa = \aleph_0$ is the Ramsey Theorem, and for $\kappa > \aleph_0$ is the Dushnik-Miller Theorem. Since P is FAC, we must have a chain of order type at least κ . ★2.3

Such results are however of little help when the poset at hand is countable, because they only tell us that the invariants are countable infinite, as expected. This justifies the use of ordinal invariants rather than cardinal ones.

2.6 WPOs as a Basis for FAC Posets

A *lexicographic sum* of posets in some family $\{P_i : i \in Q\}$ of disjoint orders *along* a poset (Q, \leq_Q) , denoted by $\sum_{i \in Q} P_i$, is defined as the order \leq on the disjoint union P of $\{P_i : i \in Q\}$ such that for all $x, y \in P$ we have $x \leq y$ iff $x, y \in P_i$ for some $i \in Q$ and $x \leq_{P_i} y$, or $x \in P_i$ and $y \in P_j$ for some $i, j \in Q$ satisfying $i <_Q j$.

The lexicographic sum of copies of P along Q is denoted by $P \cdot Q$ and called the *direct product* of P and Q . The *disjoint sum* of posets in $\{P_i : i \in Q\}$ is defined as the union of the orders \leq_{P_i} ; this is just a special case of a lexicographic sum, where the sum is taken over an antichain Q . In the case of two orders P_1, P_2 , the lexicographic sum is denoted by $P_1 \sqcup P_2$.

As a consequence of Thm. 7.3 of Abraham et al (2012) (by taking the union over all infinite cardinals κ), one obtains the following classification theorem.

Theorem 2.4 (Abraham et al). *Let \mathcal{BP} be the class of posets which are either a WPO, the reverse of a WPO, or a linear order. Let \mathcal{P} be the closure of \mathcal{BP} under lexicographic sums with index set in \mathcal{BP} and augmentation. Then \mathcal{P} is exactly the class of all FAC posets.*

We will use the classification in Thm. 2.4 to see that if we know how to calculate $w(P)$ for P an arbitrary WPO, then we can bound $w(P)$ for any FAC poset P . This in fact follows from some simple observations concerning the orders in the class \mathcal{BP} .

Lemma 2.5. (1) *If P is total, then $w(P) = 1$. In general, if all the antichains in a poset P are of length $\leq n$ for some $n < \omega$, then $w(P) \leq n$, and $w(P) = n$ in the case that there are antichains of length n .*

(2) *For any poset P , $\text{Inc}(P) = \text{Inc}(P^*)$ and hence in the case of FAC posets we have $w(P^*) = w(P)$.*

(3) *If P' is an augmentation of a FAC poset P , then $\text{Inc}(P')$ is a subtree of $\text{Inc}(P)$ and therefore $w(P') \leq w(P)$.*

(4) Let P be the lexicographic sum of posets $\{P_i : i \in L\}$ along some linear order L . Then $\text{Inc}(P) = \bigcup_{i \in L} \text{Inc}(P_i)$ and in the case of FAC posets we have $\mathbf{w}(P) = \sup_{i \in L} \mathbf{w}(P_i)$.

Proof. (1) The only non-empty sequences of antichains in a linear order P are the singleton sequences. It is clear that the resulting tree $\text{Inc}(P)$ has rank 1, by assigning the value 0 to any singleton sequence. The more general statement is proved in the same way, namely if all the antichains in a poset P are of length $< n$ for some $n < \omega$ then it suffices to define $f: \text{Inc}(P) \rightarrow n$ by letting $f(s) \stackrel{\text{def}}{=} n - |s|$.

(2), (3) Obvious.

(4) This is the same argument as in Thm. 4.1.(3).

★2.5

In conjunction with Thm. 2.4, we conclude that the problem of bounding the width of any given FAC poset is reduced to knowing how to calculate the width of WQO posets. This is the consideration of the second part of this article, starting with Sec. 4.

3 Characterisations of Ordinal Invariants

We recall in this section the known characterisations of ordinal invariants. With the method of residuals we can follow Kříž and Thomas (1990) and show that the height and maximal order types of WPOs also correspond to their maximal chain heights (Sec. 3.1) and maximal linearisation heights (Sec. 3.2), relying on results of Wolk (1967) and de Jongh and Parikh (1977) to show that these maxima are indeed attained. In a similar spirit, the width of a FAC poset is equal to its antichain rank (Sec. 3.4), an invariant studied by Abraham and Bonnet (1999)—but this time it is not necessarily attained. Finally, in Sec. 3.5 we recall an inequality relating all three invariants and shown by Kříž and Thomas (1990).

3.1 Height and Maximal Chains

Given a WF poset P , let $\mathcal{C}(P)$ denote its set of non-empty chains. Each chain C from $\mathcal{C}(P)$ is well-founded and has a rank $\mathbf{h}(C)$; we denote the supremum of these ranks by $\text{rk}_{\mathcal{C}} P \stackrel{\text{def}}{=} \sup_{C \in \mathcal{C}(P)} \mathbf{h}(C)$. As explained for example by Kříž and Thomas (1990, Thm. 4.9), we have

$$\text{rk}_{\mathcal{C}} P \leq \mathbf{h}(P) \tag{3}$$

and this can be shown, for instance, by induction on the height using the method of residuals. Indeed, (3) holds when $P = \emptyset$, and for the induction step

$$\sup_{C \in \mathcal{C}(P)} \mathbf{h}(C) \stackrel{(2)}{=} \sup_{C \in \mathcal{C}(P)} (\sup_{x \in C} \{\mathbf{h}(C_{<x}) + 1\}) \leq \sup_{x \in P} \{(\sup_{C' \in \mathcal{C}(P_{<x})} \mathbf{h}(C')) + 1\}$$

because $C_{<x}$ is a chain in $\mathcal{C}(P_{<x})$, and then by induction hypothesis (3)

$$\sup_{C \in \mathcal{C}(P)} \mathbf{h}(C) \leq \sup_{x \in P} \{\mathbf{h}(P_{<x}) + 1\} \stackrel{\omega}{=} \mathbf{h}(P).$$

Remark 3.1. The inequality in (3) can be strict. For instance, consider the forest F defined by the disjoint union $\{C_n : n \in \mathbb{N}\}$ along $(\mathbb{N}, =)$, where each C_n is a chain of height n , and add a new top element t yielding $P \stackrel{\text{def}}{=} t \hat{\cup} F$. Then P is WF (but not FAC and is thus not a WPO). Note that $\mathbf{h}(P) = \mathbf{h}(F) + 1 = \omega + 1$. However, every chain C in $\mathcal{C}(P)$ is included in $t \hat{\cup} C_n$ for some n and has height bounded by $n + 1$, while $\text{rk}_{\mathcal{C}(P)} = \omega < \mathbf{h}(P)$. ★_{3.1}

Wolk (1967, Thm. 9) further shows that, when P is a WPO, the supremum is attained, i.e. there is a chain C with $\text{rank } \mathbf{h}(C) = \text{rk}_{\mathcal{C}(P)}$. In such a case, (3) can be strengthened to

$$\max_{C \in \mathcal{C}(P)} \mathbf{h}(C) = \text{rk}_{\mathcal{C}(P)} = \mathbf{h}(P) \quad (4)$$

as can be checked by well-founded induction with

$$\mathbf{h}(P) \stackrel{\omega}{=} \sup_{x \in P} \{\mathbf{h}(P_{<x}) + 1\} \leq \sup_{x \in P} \{\mathbf{h}(C_x) + 1\} \leq \sup_{x \in P} \mathbf{h}(C_x \cup \{x\}) \leq \sup_{C \in \mathcal{C}(P)} \mathbf{h}(C)$$

where C_x is a chain of $P_{<x}$ witnessing (4) by induction hypothesis, and $C_x \cup \{x\}$ is therefore a chain in $\mathcal{C}(P)$ of height $\mathbf{h}(C_x) + 1$.

Theorem 3.2 (Wolk; Kríž and Thomas). *Let P be a WPO. Then $\mathbf{h}(P) = \text{rk}_{\mathcal{C}(P)}$ and $\max_{C \in \mathcal{C}(P)} \mathbf{h}(C)$ is the maximal height of the non-empty chains of P .*

More generally, the WPO condition in Thm. 3.2 can be relaxed using the following result proven in (Pouzet, 1979; Schmidt, 1981; Milner and Sauer, 1981).

Theorem 3.3 (Pouzet; Schmidt; Milner and Sauer). *Let P be a WF poset. Then*

- either $\text{rk}_{\mathcal{C}(P)} = \max_{C \in \mathcal{C}(P)} \mathbf{h}(C)$, i.e. there exist chains of maximal height,
- or there exists an antichain A of P such that the set of heights $\{\mathbf{h}(P_{<x}) : x \in A\}$ is infinite.

3.2 Maximal Order Types and Linearisations

A *linearisation* of a poset (P, \leq) is an augmentation $L = (P, \preceq)$ which is a total order: $x \leq y$ implies $x \preceq y$. We let $\mathcal{L}(P)$ denote the set of linearisations of P . As stated by de Jongh and Parikh (1977), a poset is a WPO if and only if all its linearisations are well-founded. De Jongh and Parikh furthermore considered the supremum $\sup_{L \in \mathcal{L}(P)} \mathbf{h}(L)$ of the order types of the linearisations of P , and showed that this supremum was attained (de Jongh and Parikh, 1977, Thm. 2.13); this is also the subject of (Blass and Gurevich, 2008, Thm. 10).

Theorem 3.4 (de Jongh and Parikh; Kříž and Thomas). *Let Q be a WQO. Then $\mathfrak{o}(Q) = \max_{L \in \mathcal{L}(Q)} \mathfrak{h}(L)$ is the maximal height of the linearisations of Q .*

3.3 Maximal Order Types and Height of Downwards-Closed Sets

A subset D of a WQO (Q, \leq) is *downwards-closed* if, for all y in D and $x \leq y$, x also belongs to D . We let $\mathcal{D}(Q)$ denote the set of downwards-closed subsets of Q . For instance, when $Q = \omega$, $\mathcal{D}(\omega)$ is isomorphic to $\omega + 1$.

It is well-known that a quasi-order Q is WQO if and only if it satisfies the descending chain condition, meaning that $(\mathcal{D}(Q), \subseteq)$ is well-founded. Therefore $\mathcal{D}(Q)$ has a rank $\mathfrak{h}(\mathcal{D}(Q))$ when Q is WQO. As shown by Blass and Gurevich (2008, Prop. 31), this can be compared to the maximal order type of Q .

Theorem 3.5 (Blass and Gurevich). *Let Q be a WQO. Then $\mathfrak{o}(Q) + 1 = \mathfrak{h}(\mathcal{D}(Q))$.*

3.4 Width and Antichain Rank

Abraham and Bonnet (1999) consider a structure similar to the tree $\text{Inc}(P)$ for FAC posets P , namely the poset $\mathcal{A}(P)$ of all non-empty antichains of P . In the case of a FAC poset, the poset $(\mathcal{A}(P), \supseteq)$ is well-founded. Let us call its height the *antichain rank* of P and denote it by $\text{rk}_{\mathcal{A}} P \stackrel{\text{def}}{=} \mathfrak{h}(\mathcal{A}(P))$; this is the smallest ordinal γ such that there is a strict order-preserving function from $\mathcal{A}(P)$ to γ .

In fact the antichain rank and the width function we study have the same values, as we now show. Thus one can reason about the width $\mathfrak{w}(P)$ by looking at the tree $\text{Inc}(P)$ or at $(\mathcal{A}(P), \supseteq)$, a different structure.

Theorem 3.6. *Let P be a FAC poset. Then $\mathfrak{w}(P) = \text{rk}_{\mathcal{A}} P$.*

Proof. Let $\gamma = \text{rk}_{\mathcal{A}} P$ and let $r: \mathcal{A}(P) \rightarrow \gamma$ be such that $S \supseteq T \implies r(S) < r(T)$ for all non-empty antichains S, T . Define $f: \text{Inc}(P) \rightarrow \gamma$ by letting for s non-empty $f(s) \stackrel{\text{def}}{=} r(S)$, where S is the set of elements of s . This function satisfies $s \triangleleft t \implies f(s) > f(t)$ and hence $\mathfrak{w}(P) \leq \text{rk}_{\mathcal{A}} P$.

Conversely, let $\gamma = \mathfrak{w}(P)$ and $f: \text{Inc}(P) \rightarrow \gamma$ be such that $s \triangleleft t \implies f(s) > f(t)$. For a non-empty antichain $S \in \mathcal{A}(P)$, observe that there exist finitely many—precisely $|S|!$ —sequences s in $\text{Inc}(P)$ with support set S . Call this set $\text{Lin}(S)$ and define $r: \mathcal{A}(P) \rightarrow \gamma$ by $r(S) \stackrel{\text{def}}{=} \min_{s \in \text{Lin}(S)} f(s)$. Consider now an antichain S with $r(S) = f(s)$ for some $s \in \text{Lin}(S)$, and an antichain T with $T \supseteq S$: then there exists an extension t of s in $\text{Lin}(T)$, which is therefore such that $f(s) > f(t)$, and hence $r(S) = f(s) > f(t) \geq r(T)$. Thus $\mathfrak{w}(P) \geq \text{rk}_{\mathcal{A}} P$. ★3.6

Remark 3.7. The width $\mathfrak{w}(P)$ is in general not attained, i.e., there might not exist any chain of antichains of height $\mathfrak{w}(P)$. First note that even when P is a WPO, $(\mathcal{A}(P), \supseteq)$ is in general not a WPO, hence Thm. 3.2 does not apply. In fact, examples of FAC

posets where the width is not attained abound. Consider indeed any FAC poset P with $\mathbf{w}(P) \geq \omega$, and any non-empty chain C in $\mathcal{C}(\mathcal{A}(P))$. As C is well-founded for \supseteq , it has a minimal element, which is an antichain $A \in \mathcal{A}(P)$ such that, for all $A' \neq A$ in C , $A' \subsetneq A$. Since P is FAC, A is finite, and C is therefore finite as well: $\mathbf{h}(C) < \omega$. ★_{3.7}

3.5 Relationship Between Width, Height and Maximal Order Type

As we have seen in the previous discussion, $\mathbf{w}(P) = \mathbf{h}(\mathcal{A}(P))$ the antichain rank (where antichains are ordered by reverse inclusion). Kříž and Thomas (1990, Thm. 4.13) proved that there is another connection between the ordinal functions discussed here and the width function.

The statement uses natural products of ordinals. Recall for this that the Cantor normal form (CNF) of an ordinal α

$$\alpha = \omega^{\alpha_0} \cdot m_0 + \cdots + \omega^{\alpha_\ell} \cdot m_\ell$$

is determined by a non-empty decreasing sequence $\alpha_0 > \alpha_1 \cdots > \alpha_\ell \geq 0$ of ordinals and a sequence of natural numbers $m_i > 0$. Cantor proved that every ordinal has a unique representation in this form. Two well-known operations can be defined based on this representation: the *natural or Hessenberg sum* $\alpha \oplus \beta$ is defined by adding the coefficients of the normal forms of α and β as though these were polynomials in ω . The *natural or Hessenberg product* $\alpha \otimes \beta$ is obtained when the normal forms of α and β are viewed as polynomials in ω and multiplied accordingly.

Theorem 3.8 (Kříž and Thomas). *For any WQO (Q, \leq) the following holds:*

$$\mathbf{w}(Q) \leq \mathbf{o}(Q) \leq \mathbf{h}(Q) \otimes \mathbf{w}(Q). \quad (5)$$

For completeness, we give a detailed proof.

Proof. For the first inequality, clearly any antichain in Q can be linearised in an arbitrary way in a linearisation of Q . So $\mathbf{w}(Q)$ is certainly bounded above by the length of the maximal such linearisation, which by Thm. 3.4 is exactly the value of $\mathbf{o}(Q)$.

For the second inequality, let $\alpha = \mathbf{w}(Q)$ and let $g : \text{Inc}(Q) \rightarrow \alpha$ be a function witnessing that. Also, let $\beta = \mathbf{h}(Q)$ and let $\rho : Q \rightarrow \beta$ be the rank function.

For any bad sequence $\langle q_0, q_1, \dots, q_n \rangle$ in Q we know that $i < j \leq n$ implies that either q_i is incomparable with q_j or $q_i > q_j$ and hence, in the latter case $\rho(q_i) > \rho(q_j)$. Fixing a bad sequence $s = \langle q_0, q_1, \dots, q_n \rangle$, consider the set

$$S_s \stackrel{\text{def}}{=} \{ \langle q_{i_0}, q_{i_1}, \dots, q_{i_m} \rangle : i_0 < i_1 \cdots < i_m = n \wedge \rho(q_{i_0}) \leq \rho(q_{i_1}) \cdots \leq \rho(q_{i_m}) \}.$$

In other words, S_s consists of subsequences of s that end with q_n and where all elements are incomparable. So for each $t \in S_s$ the value $g(t)$ is defined. We define

that $\varphi(s)$ is the minimum over all $g(t)$ for $t \in S_s$. The intuition here is that φ is an ordinal measure for the longest incomparable sequence within a bad sequence. Now we are going to combine ρ and φ into a function f defined on bad sequences. Given such a sequence $s = \langle q_0, q_1, \dots, q_n \rangle$, we let

$$f(s) \stackrel{\text{def}}{=} \langle (\rho(q_0), \varphi(\langle q_0 \rangle)), (\rho(q_1), \varphi(\langle q_0, q_1 \rangle)), \dots, (\rho(q_n), \varphi(\langle q_0, q_1, \dots, q_n \rangle)) \rangle.$$

Noticing that every non-empty subsequence of a bad sequence is bad, we see that f is a well-defined function which maps $\text{Bad}(Q)$ into the set of finite sequences from $\alpha \times \beta$. Moreover, let us notice that every sequence in the image of f is a bad sequence in $\alpha \times \beta$: if $i < j$ and $\rho(q_i) \leq \rho(q_j)$, let t be a sequence from $S_{\langle q_0, q_1, q_2, \dots, q_i \rangle}$ such that $g(t) = \varphi(\langle q_0, q_1, q_2, \dots, q_i \rangle)$. Hence t includes q_i and for every $q_k \in t$ we have $\rho(q_k) \leq \rho(q_i) \leq \rho(q_j)$. Therefore $t \frown q_j$ was taken into account when calculating $\varphi(\langle q_0, q_1, q_2, \dots, q_j \rangle)$. In particular,

$$\varphi(\langle q_0, q_1, q_2, \dots, q_j \rangle) \leq g(t \frown q_j) < g(t) = \varphi(\langle q_0, q_1, q_2, \dots, q_i \rangle). \quad (6)$$

Then $(\rho(q_i), \varphi(\langle q_0, q_1, q_2, \dots, q_i \rangle)) \not\leq (\rho(q_j), \varphi(\langle q_0, q_1, q_2, \dots, q_j \rangle))$. Another possibility when $i < j$ is that $\rho(q_i) > \rho(q_j)$ and it yields the same conclusion. We have therefore shown that $f : \text{Bad}(Q) \rightarrow \text{Bad}(\alpha \times \beta)$. Let us also convince ourselves that f is a tree homomorphism, meaning a function that preserves the strict tree order. The tree $\text{Bad}(Q)$ is ordered by initial segments, the order which we have denoted by \triangleleft . If $s \triangleleft t$, then obviously $f(s) \triangleleft f(t)$. Given that it is well known and easy to see that tree homomorphisms can only increase the rank of a tree, we have that $\mathfrak{o}(Q) \leq \mathfrak{o}(\alpha \times \beta)$. The latter, as shown by de Jongh and Parikh (1977), is equal to $\alpha \otimes \beta = \mathfrak{w}(Q) \otimes \mathfrak{h}(Q)$ (note that \otimes is commutative). $\star_{3.8}$

From Thm. 3.8 we derive a useful consequence. Recall that α is *additive* (or *multiplicative*) *principal* if $\beta, \gamma < \alpha$ implies $\beta + \gamma < \alpha$ (respectively implies $\beta \cdot \gamma < \alpha$). These implications also hold for natural sums and products.

Corollary 3.9. *Assume that $\mathfrak{o}(Q)$ is a principal multiplicative ordinal and that $\mathfrak{h}(Q) < \mathfrak{o}(Q)$. Then $\mathfrak{w}(Q) = \mathfrak{o}(Q)$.*

Proof. Assume, by way of contradiction, that $\mathfrak{w}(Q) < \mathfrak{o}(Q)$. From $\mathfrak{h}(Q) < \mathfrak{o}(Q)$ we deduce $\mathfrak{h}(Q) \otimes \mathfrak{w}(Q) < \mathfrak{o}(Q)$ (since $\mathfrak{o}(Q)$ is multiplicative principal), contradicting the inequality (5) in Thm. 3.8. Hence $\mathfrak{w}(Q) \geq \mathfrak{o}(Q)$, and necessarily $\mathfrak{w}(Q) = \mathfrak{o}(Q)$, again by (5). $\star_{3.9}$

4 Computing the Invariants of Common WQOs

We now consider WQOs obtained in various well-known ways and address the question of computing their width, and recall along the way what is known about their height and maximal order type.

In the ideal case, there would be a means of defining well-quasi-orders as the closure of some simple orders, in the ‘Hausdorff-like’ spirit of Thm. 2.4. Unfortunately,

no such result is known and indeed it is unclear which class of orders one could use as a base—for example how would one obtain Rado’s example (see Sec. 4.6) from a base of any ‘reasonable orders.’ Therefore, our study of the width of WQO orders will have to be somewhat pedestrian, concentrating on concrete situations.

4.1 Lexicographic Sums

In the case of lexicographic sums along an ordinal (defined in Sec. 2.6), we have the following result.

Lemma 4.1. *Suppose that for an ordinal α we have a family of WQOs $\{P_i : i < \alpha\}$. Then $\Sigma_{i < \alpha} P_i$ is a WQO, and:*

1. $\mathbf{o}(\Sigma_{i < \alpha} P_i) = \Sigma_{i < \alpha} \mathbf{o}(P_i)$,
2. $\mathbf{h}(\Sigma_{i < \alpha} P_i) = \Sigma_{i < \alpha} \mathbf{h}(P_i)$,
3. $\mathbf{w}(\Sigma_{i < \alpha} P_i) = \sup_{i < \alpha} \mathbf{w}(P_i)$.

Proof. First note that any infinite bad sequence in $\Sigma_{i < \alpha} P_i$ would either have an infinite projection to α or an infinite projection to some P_i , which is impossible. Hence $\Sigma_{i < \alpha} P_i$ is a WQO. Therefore the values $\mathbf{w}(\Sigma_{i < \alpha} P_i)$, $\mathbf{o}(\Sigma_{i < \alpha} P_i)$ and $\mathbf{h}(\Sigma_{i < \alpha} P_i)$ are well defined.

1. We use Thm. 3.2. Let $\alpha_i \stackrel{\text{def}}{=} \mathbf{o}(P_i)$, then $\Sigma_{i < \alpha} \alpha_i$ is isomorphic to a linearisation of $\Sigma_{i < \alpha} P_i$. Hence $\mathbf{o}(\Sigma_{i < \alpha} P_i) \geq \Sigma_{i < \alpha} \mathbf{o}(P_i)$. Suppose that L is a linearisation of $\Sigma_{i < \alpha} P_i$ (necessarily a well order), then the projection of L to each P_i is a linearisation of P_i and hence it has type $\leq \alpha_i$. This gives that the type of L is $\leq \Sigma_{i < \alpha} \alpha_i$, proving the other side of the desired inequality.
2. We use Thm. 3.4. Any chain C in $\Sigma_{i < \alpha} P_i$ can be obtained as $C = \Sigma_{i < \alpha} C_i$, where C_i is the projection of C on the coordinate i . The conclusion follows as in the case of \mathbf{o} .
3. Every non-empty sequence of incomparable elements in P must come from one and only one P_i , hence $\text{Inc}(P) = \bigcup_{i \in L} \text{Inc}(P_i)$, and therefore $\mathbf{w}(P) = \sup_{i < \alpha} \mathbf{w}(P_i)$ by Lem. 2.1. ★_{4.1}

4.2 Disjoint Sums

We also defined disjoint sums in Sec. 2.6 as sums along an antichain.

Lemma 4.2. *Suppose that P_1, P_2, \dots is a family of WQOs.*

1. $\mathbf{o}(P_1 \sqcup P_2) = \mathbf{o}(P_1) \oplus \mathbf{o}(P_2)$,
2. $\mathbf{h}(\bigsqcup_i P_i) = \sup\{\mathbf{h}(P_i)\}_i$,
3. $\mathbf{w}(P_1 \sqcup P_2) = \mathbf{w}(P_1) \oplus \mathbf{w}(P_2)$.

Proof. (1) is Thm. 3.4 from de Jongh and Parikh (1977).

(2) is clear since, for an arbitrary family P_i of WQOs, $\text{Dec}(\bigsqcup_i P_i)$ is isomorphic to $\bigsqcup_i \text{Dec}(P_i)$. We observe that, for infinite families, $\bigsqcup_i P_i$ is not WQO, but it is still well-founded hence has a well-defined height.

(3) is Lem. 1.10 from Abraham and Bonnet (1999) about antichain rank, which translates to widths thanks to Thm. 3.6. ★_{4.2}

We can apply lexicographic sums to obtain the existence of WQO posets of every width.

Corollary 4.3. *For every ordinal α , there is a WQO poset P_α such that $\mathbf{w}(P_\alpha) = \alpha$.*

Proof. The proof is by induction on α . For α finite, the conclusion is exemplified by an antichain of length α . For α a limit ordinal let us fix for each $\beta < \alpha$ a WPO P_β satisfying $\mathbf{w}(P_\beta) = \beta$. Then $\mathbf{w}(\sum_{\beta < \alpha} P_\beta) = \sup_{\beta < \alpha} \beta = \alpha$, as follows by Lem. 4.1. For $\alpha = \beta + 1$, we take $P_\alpha = P_\beta \sqcup 1$, i.e., P_β with an extra (incomparable) element added, and rely on $\mathbf{w}(Q \sqcup 1) = \mathbf{w}(Q) \oplus 1 = \mathbf{w}(Q) + 1$ shown in Lem. 4.2. ★_{4.3}

4.3 Direct Products

Direct products are again a particular case of lexicographic sums along a poset Q , this time of the same poset P . While the cases of \mathbf{o} and \mathbf{h} are mostly folklore, the width of $P \cdot Q$ is not so easily understood, and its computation in Lem. 1.11 from Abraham and Bonnet (1999) uses the notion of *Heisenberg products* $\alpha \odot \beta$, defined for any ordinal α by induction on the ordinal β :

$$\alpha \odot 0 \stackrel{\text{def}}{=} 0, \quad \alpha \odot (\beta + 1) \stackrel{\text{def}}{=} (\alpha \odot \beta) \oplus \alpha, \quad \alpha \odot \lambda \stackrel{\text{def}}{=} \sup\{(\alpha \odot \gamma) + 1 : \gamma < \lambda\}$$

where λ is a limit ordinal. Note that this differs from the natural product, and is not commutative: $2 \odot \omega = \omega$ but $\omega \odot 2 = \omega \cdot 2$.

Lemma 4.4 (Abraham and Bonnet). *Suppose that P and Q are two WPOs.*

1. $\mathbf{o}(P \cdot Q) = \mathbf{o}(P) \cdot \mathbf{o}(Q)$,
2. $\mathbf{h}(P \cdot Q) = \mathbf{h}(P) \cdot \mathbf{h}(Q)$,
3. $\mathbf{w}(P \cdot Q) = \mathbf{w}(P) \odot \mathbf{w}(Q)$.

4.4 Cartesian Products

The next simplest operation on WQOs is their Cartesian product. It turns out that the simplicity of the operation is deceptive and that the height and, especially, the width of a product $P \times Q$ are not as simple as we would like. As a consequence, this section only provides partial results and is unexpectedly long.

To recall, the product order $P \times Q$ of two partial orders is defined on the pairs (p, q) with $p \in P$ and $q \in Q$ so that $(p, q) \leq (p', q')$ iff $p \leq_P p'$ and $q \leq_Q q'$. It is easy to check and well known that product of WQOs is WQO and similarly for FAC and WF orders.

The formula for calculating $\mathfrak{o}(P \times Q)$ is still simple. It was first established by de Jongh and Parikh (1977, Thm. 3.5); see also (Blass and Gurevich, 2008, Thm. 6).

Lemma 4.5 (de Jongh and Parikh). *Suppose that P and Q are two WQOs. Then $\mathfrak{o}(P \times Q) = \mathfrak{o}(P) \otimes \mathfrak{o}(Q)$.*

The question of the height of products is also well studied and a complete answer appears in (Abraham, 1987), where it is stated that the theorem is well known. The following statement is a reformulation of Lem. 1.8 of Abraham (1987).

Lemma 4.6 (Abraham; folklore). *If $\rho_P : P \rightarrow \mathbf{h}(P)$ and $\rho_Q : Q \rightarrow \mathbf{h}(Q)$ are the rank functions of the well-founded posets P and Q , then the rank function ρ on $P \times Q$ is given by $\rho(x, y) = \rho_P(x) \oplus \rho_Q(y)$. In particular,*

$$\mathbf{h}(P \times Q) = \sup\{\alpha \oplus \beta + 1 : \alpha < \mathbf{h}(P) \wedge \beta < \mathbf{h}(Q)\}.$$

We recall that for any two ordinals α and β we have $\sup_{\alpha' < \alpha, \beta' < \beta} \alpha' \oplus \beta' + 1 < \alpha \oplus \beta$ (see e.g. Abraham and Bonnet, 1999, p. 55), thus the statement in Thm. 4.6 cannot be easily simplified.

Remark 4.7 (Height of products of finite ordinals). The very nice general proof of Abraham (1987, Lem. 1.8) can be done in an even more visual way in the case of finite ordinals. Let $P = n_1 \times \cdots \times n_k$ for some finite $n_1, \dots, n_k \in \omega$; then $\mathbf{h}(P) = n_1 + \cdots + n_k + 1 - k$.

Indeed, we observe that any chain $\mathbf{a}_1 <_P \cdots <_P \mathbf{a}_\ell$ in P leads to a strictly increasing $|\mathbf{a}_1| < \cdots < |\mathbf{a}_\ell|$, where by $|\mathbf{a}|$ we denote the sum of the numbers in \mathbf{a} . Since $|\mathbf{a}_\ell|$ is at most $\sum_i (n_i - 1) = (\sum_i n_i) - k$ and since $|\mathbf{a}_1|$ is at least 0, the longest chain has length $1 + \sum_i n_i - k$. Furthermore it is easy to build a witness for this length. We conclude by invoking Thm. 3.4 which states that for any WPO P , $\mathbf{h}(P)$ is the length of the longest chain in P . ★_{4.7}

Having dealt with \mathbf{h} and \mathfrak{o} , we are left with \mathbf{w} . Here we cannot hope to have a uniform formula expressing $\mathbf{w}(P \times Q)$ as a function of $\mathbf{w}(P)$ and $\mathbf{w}(Q)$. Indeed, already in the case of ordinals one always has $\mathbf{w}(\alpha) = \mathbf{w}(\beta) = 1$, while $\mathbf{w}(\alpha \times \beta)$ has quite a complex form, as we are going to see next.

4.4.1 Products of Ordinals

Probably the simplest example of WQO which is not actually an ordinal, is provided by the product of two ordinals. Thanks to Thm. 3.6, we can translate results of Abraham (1987), Section 3 to give a recursive formula which completely characterises $\mathbf{w}(\alpha \times \beta)$ for α, β ordinals. We shall sketch how this is done.

First note that if one of α, β is a finite ordinal n , say $\alpha = n$, then we have $\mathbf{w}(n \times \beta) = \min\{n, \beta\}$. The next case to consider is that of successor ordinals, which is taken care by the following Thm. 4.8. Abraham proved this theorem using the method of residuals and induction, we offer an alternative proof using the rank of the tree Inc.

Theorem 4.8 (Abraham). *For any ordinals α, β with α infinite, we have $\mathbf{w}(\alpha \times (\beta + 1)) = \mathbf{w}(\alpha \times \beta) + 1$.*

The proof is provided by the next two lemmas.

Lemma 4.9. $\mathbf{w}(\alpha \times (\beta + 1)) \leq \mathbf{w}(\alpha \times \beta) + 1$ for any ordinals α, β .

Proof. Write I for $\text{Inc}(\alpha \times (\beta + 1))$ and I' for $\text{Inc}(\alpha \times \beta)$. Any sequence $s = \langle p_1, \dots, p_\ell \rangle$ which is in I , is either in I' or contains a single pair of the form $p_i = (a, \beta)$, with $a < \alpha$. In the latter case we write s' for s with p_i removed. Note that s' is in I' (except when s has length 1). Let $\rho' : I' \rightarrow \text{rank}(I') = \mathbf{w}(\alpha \times \beta)$ be a ranking function for I' and define $\rho : I \rightarrow ON$ via

$$\rho(s) \stackrel{\text{def}}{=} \begin{cases} \rho'(s) + 1 & \text{if } s \in I', \\ \rho'(s') & \text{if } s \notin I' \text{ and } |s| > 1, \\ \text{rank}(I') & \text{otherwise.} \end{cases}$$

One easily checks that ρ is anti-monotone. For this assume $s \triangleleft t$: (1) if both s and t are in I' , monotonicity is inherited from ρ' ; (2) if none are in I' then $s' \triangleleft t'$ (or s' is empty) and again monotonicity is inherited (or $\rho(s) = \text{rank}(I') > \rho'(t') = \rho(t)$); (3) if s is in I' and t is not then $s \trianglelefteq t'$, entailing $\rho'(s) \geq \rho'(t')$ so that $\rho(s) = \rho'(s) + 1 > \rho'(t') = \rho(t)$.

In conclusion ρ , having values in $\mathbf{w}(\alpha \times \beta) + 1$, witnesses the assertion of the lemma. ★_{4.9}

Lemma 4.10. *If α is infinite then $\mathbf{w}(\alpha \times (\beta + 1)) \geq \mathbf{w}(\alpha \times \beta) + 1$ for any β .*

Proof. Write I for $\text{Inc}(\alpha \times \beta)$. Any $s \in I$ has the form $s = \langle (a_1, b_1), \dots, (a_\ell, b_\ell) \rangle$. We write s_+ for the sequence $\langle (a_1 + 1, b_1), \dots, (a_\ell + 1, b_\ell) \rangle$ and observe that it is still a sequence over $\alpha \times \beta$ since α is infinite, and that its elements form an antichain (since the elements of s did). Let now s'_+ be $r \frown s_+$ where $r = \langle (0, \beta) \rangle$: the prepended element is not comparable with any element of s_+ so that s'_+ is an antichain and $s'_+ \trianglelefteq t'_+$ iff $s_+ \trianglelefteq t_+$ iff $s \trianglelefteq t$. Write I'_+ for $\{s'_+ \mid s \in I\} \cup \{r\}$. This is a tree made of a root glued below a tree isomorphic to I . Hence $\text{rank}(I'_+) = \text{rank}(I) + 1$. On the other hand, I'_+ is a substructure of $\text{Inc}(\alpha \times (\beta + 1))$ hence $\mathbf{w}(\alpha \times (\beta + 1)) \geq \text{rank}(I'_+)$. ★_{4.10}

With Thm. 4.8 in hand, the remaining case is to compute $\mathbf{w}(\alpha \times \beta)$ when α, β are limit ordinals. This translates into saying that $\alpha = \omega\alpha'$ and $\beta = \omega\beta'$ for some $\alpha', \beta' > 0$. A recursive formula describing the weight of this product is the main theorem of Section 3 of Abraham (1987), which we now quote. It is proved using a complex application of the method of residuals and induction.

Theorem 4.11 (Abraham). *Suppose that α and β are given in their Cantor normal forms $\alpha = \omega^{\alpha_0} \cdot m_0 + \rho$, $\beta = \omega^{\beta_0} \cdot n_0 + \sigma$, where $\omega^{\alpha_0} \cdot m_0$ and $\omega^{\beta_0} \cdot n_0$ are the leading terms and ρ and σ are the remaining terms of the Cantor normal forms of α and β respectively. Then if $\alpha = 1$, we have $\mathbf{w}(\omega \times \omega\beta) = \omega\beta$, and in general*

$$\mathbf{w}(\omega\alpha \times \omega\beta) = \omega\omega^{\alpha_0 \oplus \beta_0} \cdot (m_0 + n_0 - 1) \oplus \mathbf{w}(\omega\omega^{\alpha_0} \times \omega\sigma) \oplus \mathbf{w}(\omega\omega^{\beta_0} \times \omega\rho).$$

It would be interesting to have a closed rather than a recursive formula for the width of the product of two ordinals. However, the formula does give us a closed form of values of the weight of the product of two ordinals with only one term in the Cantor normal form, as we now remark. Here m, n are finite ordinals ≥ 1 .

1. If $k, \ell < \omega$ then we have

$$\mathbf{w}(\omega^{1+k} \cdot m \times \omega^{1+\ell} \cdot n) = \mathbf{w}(\omega(\omega^k \cdot m) \times \omega(\omega^\ell \cdot n)) = \omega^{k+\ell-1} \cdot (m+n-1).$$

2. (example 3.4 (3) from Abraham (1987)) If $\alpha, \beta \geq \omega$ then $1 + \alpha = \alpha$ and $1 + \beta = \beta$, so

$$\mathbf{w}(\omega^\alpha \cdot m \times \omega^\beta \cdot n) = \mathbf{w}(\omega(\omega^\alpha \cdot m) \times \omega(\omega^\beta \cdot n)) = \omega^{\alpha \oplus \beta} \cdot (m+n-1).$$

3. If $\alpha \geq \omega$ and $k < \omega$ then $\mathbf{w}(\omega^\alpha \cdot m \times \omega^{1+k} \cdot n) = \omega^{\alpha+k} \cdot (m+n-1)$.

Let us mention one more result derivable from Thm. 4.11.

Lemma 4.12 (Abraham). $\mathbf{w}(\omega \times \alpha) = \alpha$ for any ordinal α .

Proof. By induction on α . If α is a limit, we write it $\alpha = \omega\alpha' = \omega(\omega^{\alpha_0} \cdot m_0 + \dots + \omega^{\alpha_\ell} \cdot m_\ell)$. Now Thm. 4.11 yields $\mathbf{w}(\omega \times \omega\alpha') = \omega\omega^{\alpha_0} \cdot m_0 \oplus \dots \oplus \omega\omega^{\alpha_\ell} \cdot m_\ell = \alpha$. If α is a successor, we use Lem. 4.9 and 4.10. ★4.12

4.4.2 Finite Products and Transferable Orders

Since the width of the product of two ordinals is understood, we can approach the general question of the width of products of two or a finite number of WQO posets P_i by reducing it to the width of some product of ordinals. Using that strategy, we give a lower bound to $\mathbf{w}(\prod_{i \leq n} P_i)$.

Theorem 4.13. For any WQO posets $P_0, P_1 \dots P_n$, $\mathbf{w}(\prod_{i \leq n} P_i) \geq \mathbf{w}(\prod_{i \leq n} \mathbf{h}(P_i))$.

The proof follows directly from a simple lemma, which is of independent interest:

Lemma 4.14. Suppose that $P_0, P_1 \dots P_n$ are WQO posets. Then $\prod_{i \leq n} \mathbf{h}(P_i)$ embeds into $\prod_{i \leq n} P_i$ as a substructure.

Proof. We use Thm. 3.2 and pick, in each P_i , a chain C_i in P_i that has order type $\mathbf{h}(C_i) = \mathbf{h}(P_i)$. Then $\prod_{i \leq n} C_i$ is an induced suborder of $\prod_{i \leq n} \mathbf{h}(P_i)$ which is isomorphic to $\prod_{i \leq n} \mathbf{h}(P_i)$. ★4.14

Now we shall isolate a special class of orders for which it will be possible to calculate certain widths of products. Let us write $\downarrow x$ for the downwards-closure of an element x , i.e., for $\{y : x \leq y\}$.

Definition 4.15. A FAC partial order P belongs to the class \mathcal{T} of *transferable orders* if $\mathbf{w}(P \setminus (\downarrow x_1 \cup \dots \cup \downarrow x_n)) = \mathbf{w}(P)$ for any (finitely many) elements $x_1, \dots, x_n \in P$.

Theorem 4.16. *Suppose that P is a WQO transferable poset and δ is an ordinal. Then $\mathbf{w}(P \times \delta) \geq \mathbf{w}(P) \cdot \delta$.*

Proof. Write γ for $\mathbf{w}(P)$: we prove that Player 2 has a winning strategy, denoted $\sigma_{P' \times \delta, \alpha}$, for each game $G_{P' \times \delta, \alpha}$ where P' is some $P \setminus (\downarrow y_1 \cup \dots \cup \downarrow y_n)$ and $\alpha \leq \gamma \cdot \delta$.

The proof is by induction on δ .

If $\delta = 0$ then $\alpha = 0$ and Player 1 loses immediately.

If $\delta = \lambda$ is a limit, the strategy for Player 2 depends on Player 1's first move. Say it is $\alpha' < \alpha \leq \gamma \cdot \delta$. Then $\alpha' < \gamma \cdot \delta$ means that $\alpha' < \gamma \cdot \delta'$ for some $\delta' < \delta$. Player 2 chooses one such δ' and now applies $\sigma_{P' \times \delta', \alpha' + 1}$ (which exists and is winning by the induction hypothesis) for the whole game. Note that a strategy for a substructure $P' \times \delta'$ of the original $P' \times \delta$ will lead to moves that are legal in the original game. Also note that $\alpha' + 1 \leq \gamma \cdot \delta'$.

If $\delta = \varepsilon + 1$ is a successor then Player 2 answers each move $\alpha_1, \dots, \alpha_m$ played by Player 1 by writing it in the form $\alpha_i = \gamma \cdot \delta_i + \beta_i$ with $\beta_i < \gamma$. Note that $\delta_i < \delta$. If $\delta_1 = \dots = \delta_m = \varepsilon$, note that $\beta_1 > \beta_2 > \dots > \beta_m$. Let Player 2 play (x_m, ε) where x_m is $\sigma_{P', \gamma}$ applied on β_1, \dots, β_m (that strategy exists and is winning since P is transferable and has width γ). If $\delta_m < \varepsilon$ then Player 2 switches strategy and now uses $\sigma_{P'' \times \varepsilon, \gamma \cdot \varepsilon}$ as if a new game was starting with α_m as Player 1's first move, and for $P'' = P' \setminus (\downarrow x_1 \cup \dots \cup \downarrow x_{m-1})$. By the induction hypothesis, Player 2 will win by producing a sequence S'' in $P'' \times \varepsilon$. These moves are legal since $(x_1, \varepsilon) \dots (x_{m-1}, \varepsilon) \frown S''$ is an antichain in $P' \times (\varepsilon + 1)$. ★_{4.16}

In order to use Thm. 4.16, we need actual instances of transferable orders.

Lemma 4.17. *For any $1 \leq \alpha_1, \dots, \alpha_n$, the order $P = \omega^{\alpha_1} \times \dots \times \omega^{\alpha_n}$ is transferable.*

Proof. Since each ω^{α_i} is additive principal, $P \setminus (\downarrow x_1 \cup \dots \cup \downarrow x_m)$ contains an isomorphic copy of P for any finite sequence x_1, \dots, x_m of elements of P . ★_{4.17}

Theorem 4.18. *Let P be a transferable WPO poset.*

1. *Suppose that $1 \leq m < \omega$. Then $\mathbf{w}(P) \cdot m \leq \mathbf{w}(P \times m) \leq \mathbf{w}(P) \otimes m$.*
2. *If $\mathbf{w}(P) = \omega^\gamma$ for some γ , then $\mathbf{w}(P \times m) = \mathbf{w}(P) \cdot m$ (Note that this applies to any P which is the product of the form $\omega^\alpha \times \omega^\beta$, see the examples after Thm. 4.11).*
3. *$\mathbf{w}(\omega \times \omega \times \omega) = \omega^2$.*

An easy way to provide an upper bound needed in the proof of Thm. 4.18 is given by the following observation:

Lemma 4.19. *For any FAC poset P and $1 \leq m < \omega$, $\mathbf{w}(P \times m) \leq \mathbf{w}(P) \otimes m$.*

Proof. We just need to remark that $P \times m$ is an augmentation of the perpendicular sum $\sqcup_{i < m} P$ and then apply Lem. 4.2. ★4.19

Proof (of Thm. 4.18). (1) We get $\mathbf{w}(P \times m) \geq \mathbf{w}(P) \cdot m$ from Thm. 4.16. We get $\mathbf{w}(P \times m) \leq \mathbf{w}(P) \otimes m$ from Lem. 4.19.

(2) This follows because $\omega^\gamma \otimes m = \omega^\gamma \cdot m$.

(3) Let $P = \omega \times \omega$, hence we know that $\mathbf{w}(P) = \omega$. Since any $P \times m$ is a substructure of $P \times \omega$, we clearly have that $\mathbf{w}(P \times \omega) \geq \sup_{m < \omega} \mathbf{w}(P \times m) = \sup_{m < \omega} \omega \cdot m = \omega^2$. Let us now give a proof using games that $\mathbf{w}(P \times \omega) \leq \omega^2$. It suffices to give a winning strategy to Player 1 in the game $G_{P \times \omega, \gamma}$ for any ordinal $\gamma > \omega^2$.

So, given such a γ , Player 1 starts the game by choosing as his first move the ordinal ω^2 . Player 2 has to answer by choosing an element x in $P \times \omega$, say an element (p, m) with $p = (k, \ell)$. Now notice that any element of $P \times \omega$ that is incompatible with (p, m) is either an element of $P \times m$ or of the form (q, n) for some $q \leq p$ in $\omega \times \omega$, or is of the form (r, i) for some r which is incompatible with p in $\omega \times \omega$. Therefore, any next step of Player 2 has to be in an order P' which is isomorphic to an augmentation of a substructure of the disjoint union of the form

$$P \times m \sqcup [(k+1) \times (\ell+1)] \times \omega \sqcup [(k+1) \times \omega] \times \omega \sqcup [(\ell+1) \times \omega] \times \omega. \quad (7)$$

It now suffices for Player 1 to find an ordinal $o < \omega^2$ satisfying $o > \mathbf{w}(P')$ as the game will then be transferred to $G_{P', o}$, where Player 1 has a winning strategy. As ω^2 is closed under \oplus , it suffices to show that each of the orders appearing in equation (7) has weight $< \omega^2$. This is the case for $P \times m$ by (2). We have that $\mathbf{w}([(k+1) \times (\ell+1)] \times \omega) = \mathbf{w}((k+1) \times [(\ell+1) \times \omega])$, which by applying Lem. 4.19 is $\leq (\ell+1) \cdot (k+1)$. For $[(k+1) \times \omega] \times \omega$, we apply Lem. 4.19 to $\omega \times \omega$, to obtain $\mathbf{w}([(k+1) \times \omega] \times \omega) \leq \omega \cdot (k+1)$ and similarly $\mathbf{w}([(\ell+1) \times \omega] \times \omega) \leq \omega \cdot (\ell+1)$. ★4.18

4.5 Finite Multisets, Sequences, and Trees

Well-quasi-orders are also preserved by building multisets, sequences, and trees with WQO labels, together with suitable embedding relations.

Finite sequences in $Q^{<\omega}$ are compared by the *subsequence embedding* ordering defined by $s = \langle x_0, \dots, x_{n-1} \rangle \leq_* s' = \langle x'_0, \dots, x'_{p-1} \rangle$ if there exists $f: n \rightarrow p$ strictly monotone such that $x_i \leq x'_{f(i)}$ in Q for all $i \in n$. The fact that $(Q^{<\omega}, \leq_*)$ is WQO when Q is WQO was first shown by Higman (1952).

Given a WQO (Q, \leq) , a *finite multiset* over Q is a function m from $Q \rightarrow \mathbb{N}$ with finite support, i.e. $m(x) > 0$ for finitely many $x \in Q$. Equivalently, a finite multiset is a finite sequence m in $Q^{<\omega}$ where the order is irrelevant, and can be noted as a ‘set with repetitions’ $m = \{x_1, \dots, x_n\}$; we denote by $M(Q)$ the set of finite multisets over Q . The *multiset embedding* ordering is then defined by $m = \{x_0, \dots, x_{n-1}\} \leq_\diamond m' = \{x'_0, \dots, x'_{p-1}\}$ if there exists an injective function $f: n \rightarrow p$ with $x_i \leq x'_{f(i)}$ in Q for all $i \in n$. As a consequence of $(Q^{<\omega}, \leq_*)$ being WQO, $(M(Q), \leq_\diamond)$ is also WQO when Q is.

Finally, a (rooted, ordered) *finite tree* t over Q is either a leaf $x()$ for some $x \in Q$, or a term $x(t_1, \dots, t_n)$ for some $n > 0$, $x \in Q$, and t_1, \dots, t_n trees over Q . A tree has arity b if we bound n by b in this definition. We let $T(Q)$ denote the set of finite trees over Q . The *homeomorphic tree embedding* ordering is defined by $t = x(t_1, \dots, t_n) \leq_T t' = x'(t'_1, \dots, t'_p)$ (where $n, p \geq 0$) if at least one the following cases occurs:

- $t \leq_T t'_j$ for some $1 \leq j \leq p$, or
- $x \leq x'$ in Q and $t_1 \cdots t_n \leq_* t'_1 \cdots t'_p$ for the subsequence embedding relation on $T(Q)$.

The fact that $(T(Q), \leq_T)$ is WQO when Q is WQO was first shown by Higman (1952) for trees of bounded arity, before Kruskal (1960) proved it in the general case. Note that it implies $(Q^{<\omega}, \leq_*)$ being WQO for the special case of trees of arity 1.

4.5.1 Maximal Order Types

The maximal order types of $M(Q)$, $Q^{<\omega}$, and $T(Q)$ have been studied by Weiermann (2009) and Schmidt (1979); see also Van der Meeren (2015, Sec. 1.2) for a nice exposition of these results.

For finite multisets with embedding, we need some additional notations. For an ordinal α with Cantor normal form $\omega^{\alpha_1} + \dots + \omega^{\alpha_n}$ where $\alpha(P) \geq \alpha_1 \geq \dots \geq \alpha_n$, we let

$$\widehat{\alpha} \stackrel{\text{def}}{=} \omega^{\alpha_1'} + \dots + \omega^{\alpha_n'} \quad (8)$$

where α' is $\alpha + 1$ when α is an epsilon number, i.e. when $\omega^\alpha = \alpha$, and is just α otherwise.

The following is (Weiermann, 2009, Thm. 2), with a corrected proof due to Van der Meeren et al (2015, Thm. 5).

Theorem 4.20 (Weiermann). *Let Q be a WQO. Then $\mathfrak{o}(M(Q)) = \omega^{\widehat{\mathfrak{o}(Q)}}$.*

Thus, for $\mathfrak{o}(Q) < \varepsilon_0$, one has simply $\mathfrak{o}(M(Q)) = \omega^{\mathfrak{o}(Q)}$.

For finite sequences with subsequence embedding, we recall the following result by Schmidt (1979).

Theorem 4.21 (Schmidt). *Let Q be a WQO. Then*

$$\mathfrak{o}(Q^{<\omega}) = \begin{cases} \omega^{\mathfrak{o}(Q)-1} & \text{if } \mathfrak{o}(Q) \text{ is finite,} \\ \omega^{\mathfrak{o}(Q)+1} & \text{if } \mathfrak{o}(Q) = \varepsilon + n \text{ for } \varepsilon \text{ an epsilon number and } n \text{ finite,} \\ \omega^{\mathfrak{o}(Q)} & \text{otherwise.} \end{cases}$$

The case of finite trees is actually a particular case of the results of Schmidt (1979) on embeddings in structured trees. Her results were originally stated using Schütte's Klammer symbols, but can be translated in terms of the ϑ functions of

Rathjen and Weiermann (1993). Defining such ordinal notation systems is beyond the scope of this chapter; it suffices to say for our results that the ordinals at hand are going to be principal multiplicative.

Theorem 4.22 (Schmidt). *Let Q be a WQO. Then $o(T(Q)) = \vartheta(\Omega^\omega \cdot o(Q))$.*

4.5.2 Heights

For a WQO Q we define $\mathbf{h}^*(Q)$ as

$$\mathbf{h}^*(Q) \stackrel{\text{def}}{=} \begin{cases} \mathbf{h}(Q) & \text{if } \mathbf{h}(Q) \text{ is additive principal } \geq \omega, \\ \mathbf{h}(Q) \cdot \omega & \text{otherwise.} \end{cases} \quad (9)$$

We are going to show that the heights of finite multisets, finite sequences, and finite trees over Q is the same, namely $\mathbf{h}^*(Q)$.

Theorem 4.23. *Let Q be a WF poset. Then $\mathbf{h}(M(Q)) = \mathbf{h}(Q^{<\omega}) = \mathbf{h}(T(Q)) = \mathbf{h}^*(Q)$.*

Since obviously $\mathbf{h}(M(Q)) \leq \mathbf{h}(Q^{<\omega}) \leq \mathbf{h}(T(Q))$, the claim is a consequence of lemmata 4.24 and 4.26 below.

Lemma 4.24. $\mathbf{h}(T(Q)) \leq \mathbf{h}^*(Q)$.

Proof. Consider a strictly decreasing sequence $x_0 >_T x_1 >_T \dots$ in $T(Q)$, where each x_i is a finite tree over Q . Necessarily these finite trees have a nonincreasing number of nodes: $|x_0| \geq |x_1| \geq \dots$. If we add a new minimal element \perp below Q , we can transform any x_i by padding it with some \perp 's so that now the resulting x'_i has the same shape and size as x_0 . Let us use $1 + Q$ instead of $\{\perp\} + Q$ so that the new trees belong to $T(1 + Q)$, have all the same shape, and form a strictly decreasing sequence. This construction is in fact an order-reflection from $\text{Dec}(T(Q))$ to $\text{Dec}(\bigsqcup_{n < \omega} (1 + Q)^n)$, from which we get

$$\mathbf{h}(T(Q)) \leq \mathbf{h}\left(\bigsqcup_{n < \omega} (1 + Q)^n\right) = \sup_{n < \omega} \mathbf{h}([1 + Q]^n), \quad (10)$$

using Lem. 4.2.(2) for the last equality. For $n < \omega$, one has

$$\mathbf{h}([1 + Q]^n) = \sup\{(\alpha \otimes n) + 1 : \alpha < 1 + \mathbf{h}(Q)\}, \quad (11)$$

using lemmata 4.1.(2) and 4.6.

If $\mathbf{h}(Q) \leq 1$, $\mathbf{h}(T(Q)) = \mathbf{h}(Q) \cdot \omega = \mathbf{h}^*(Q)$ obviously.

For $\mathbf{h}(Q) > 1$, and thanks to (10) and (11), it is sufficient to show that $\alpha \otimes n + 1 \leq \mathbf{h}^*(Q)$ for all $n < \omega$ and all $\alpha < 1 + \mathbf{h}(Q)$. We consider two cases:

1. If $\mathbf{h}(Q) \geq \omega$ is additive principal, $\alpha < 1 + \mathbf{h}(Q) = \mathbf{h}(Q)$ entails $\alpha \otimes n < \mathbf{h}(Q)$ thus $\alpha \otimes n + 1 < \mathbf{h}(Q) = \mathbf{h}^*(Q)$.

2. Otherwise the CNF for $\mathbf{h}(Q)$ is $\sum_{i=1}^m \omega^{\alpha_i}$ with $m > 1$. Then $\alpha < 1 + \mathbf{h}(Q)$ implies $\alpha \leq \omega^{\alpha_1} \cdot m$, thus $\alpha \otimes n + 1 \leq \omega^{\alpha_1} \cdot m \cdot n + 1 \leq \omega^{\alpha_1+1} = \mathbf{h}(Q) \cdot \omega = \mathbf{h}^*(Q)$. $\star_{4.18}$

Let us write $M_n(Q)$ for the restriction of $M(Q)$ to multisets of size n .

Lemma 4.25. $\mathbf{h}(M_n(Q)) \geq \mathbf{h}(Q^n)$.

Proof. With $\mathbf{x} = \langle x_1, \dots, x_n \rangle \in Q^n$ we associate the multiset $M_{\mathbf{x}} = \{x_1, \dots, x_n\}$. Obviously $\mathbf{x} <_{\times} \mathbf{y}$ implies $M_{\mathbf{x}} \leq_{\circ} M_{\mathbf{y}}$. We further claim that $M_{\mathbf{y}} \not\leq_{\circ} M_{\mathbf{x}}$. Indeed, assume by way of contradiction that $M_{\mathbf{y}} \leq_{\circ} M_{\mathbf{x}}$. Then there is a permutation f of $\{1, \dots, n\}$ such that $y_i \leq_Q x_{f(i)}$ for all $i = 1, \dots, n$. From $\mathbf{x} \leq_{\times} \mathbf{y}$, we get

$$x_i \leq_Q y_i \leq_Q x_{f(i)} \leq_Q y_{f(i)} \leq x_{f(f(i))} \leq y_{f(f(i))} \leq_Q \dots \leq_Q x_{f^k(i)} \leq_Q y_{f^k(i)} \leq_Q \dots$$

So that for all j in the f -orbit of i , $x_j \equiv_Q x_i \equiv_Q y_j$, entailing $\mathbf{y} \equiv_{\times} \mathbf{x}$ which contradicts the assumption $\mathbf{x} <_{\times} \mathbf{y}$.

We have thus exhibited a mapping from Q^n to $M_n(Q)$ that will map chains to chains. Hence $\mathbf{h}(Q^n) \leq \mathbf{h}(M_n(Q))$. $\star_{4.24}$

Lemma 4.26. $\mathbf{h}(M(Q)) \geq \mathbf{h}^*(Q)$.

Proof. The result is clear in cases where $\mathbf{h}^*(Q) = \mathbf{h}(Q)$ and when $\mathbf{h}(Q) = 1$ entailing $\mathbf{h}(M(Q)) = \omega = \mathbf{h}^*(Q)$. So let us assume that $\mathbf{h}(Q)$ is not additive principal and has a CNF $\sum_{i=1}^m \omega^{\alpha_i}$ with $m > 1$. Thus $\mathbf{h}^*(Q) = \mathbf{h}(Q) \cdot \omega = \omega^{\alpha_1+1}$. Since by Lem. 4.6, for $0 < n < \omega$, $\mathbf{h}(Q^n) = \sup\{\alpha \otimes n + 1 : \alpha < \mathbf{h}(Q)\}$, we deduce $\mathbf{h}(Q^n) \geq \omega^{\alpha_1} \cdot n + 1$. Since $M_n(Q)$ is a substructure of $M(Q)$, and using Lem. 4.25, we deduce

$$\mathbf{h}(M(Q)) \geq \mathbf{h}(M_n(Q)) \geq \mathbf{h}(Q^n) \geq \omega^{\alpha_1} \cdot n + 1$$

for all $0 < n < \omega$, hence

$$\mathbf{h}(M(Q)) \geq \sup_{n < \omega} \omega^{\alpha_1} \cdot n + 1 = \omega^{\alpha_1} \cdot \omega = \mathbf{h}^*(Q). \quad \star_{4.26}$$

4.5.3 Widths

The previous analyses of the maximal order types and heights of $M(Q)$, $Q^{<\omega}$, and $T(Q)$ allow us to apply the correspondence between \mathbf{o} , \mathbf{h} , and \mathbf{w} shown by Kříž and Thomas (1990, Thm. 4.13), in particular its consequence spelled out in Cor. 3.9.

Theorem 4.27. *Let Q be a WQO. Then $\mathbf{w}(Q^\dagger) = \mathbf{o}(Q^\dagger)$ where Q^\dagger can be $T(Q)$, or $Q^{<\omega}$ when $\mathbf{o}(Q) > 1$, or $M(Q)$ when $\mathbf{o}(Q) > 1$ is a principal additive ordinal.*

Proof. First observe that $\mathbf{h}^*(Q) \leq \mathbf{h}(Q) \cdot \omega \leq \mathbf{o}(Q) \cdot \omega < \mathbf{o}(Q^\dagger)$ when Q^\dagger is $T(Q)$ (by Thm. 4.22), $Q^{<\omega}$ with $\mathbf{o}(Q) > 1$ (by Thm. 4.21), or $M(Q)$ with $\mathbf{o}(Q) > 1$ (by Thm. 4.20). Furthermore, when Q^\dagger is $T(Q)$ or $Q^{<\omega}$, and when it is $M(Q)$ with $\mathbf{o}(Q)$ a principal additive ordinal, $\mathbf{o}(Q^\dagger)$ is a principal multiplicative ordinal. Thus Cor. 3.9 shows that $\mathbf{w}(Q^\dagger) = \mathbf{o}(Q^\dagger)$. $\star_{4.27}$

The assumptions in Thm. 4.27 seem necessary. For instance, if $Q = 1$, then $M(1)$ is isomorphic to $1^{<\omega}$ and ω , with height ω and width 1. If $A_3 = 1 \sqcup 1 \sqcup 1$ is an antichain with three elements, then $M(A_3)$ is isomorphic with $\omega \times \omega \times \omega$, $\mathbf{h}(M(A_3)) = \omega$ by Lem. 4.6 or Thm. 4.23, $\mathbf{o}(M(A_3)) = \omega^3$ by Lem. 4.5, and $\mathbf{w}(M(A_3)) = \omega^2$ by Thm. 4.18.(3).

4.6 Infinite Products and Rado's Structure

One may wonder what happens in the case of infinite products. We remind the reader that the property of being WQO is in general not preserved by infinite products. The classical example for this was provided by Rado in Rado (1954), who defined what we call the *Rado structure*, denoted (R, \leq) :² Rado's order is given as a structure on $\omega \times \omega$ where we define

$$(a, b) \leq (a', b') \text{ if } [a = a' \text{ and } b \leq b'] \text{ or } b < a'.$$

The definition of BQOs was motivated by trying to find a property stronger than WQO which is preserved by infinite products, so in particular Rado's example is not a BQO (see Milner, 1985, Thm. 1.11 and 2.22).

We can use the method of residuals and other tools described in previous sections to compute.

$$\mathbf{o}(R) = \omega^2, \quad \mathbf{h}(R) = \omega, \quad \mathbf{w}(R) = \omega, \quad (12)$$

which gives the same ordinal invariants as those of the product $\omega \times \omega$, even though they are not isomorphic, and moreover $\omega \times \omega$ is a BQO (since the notion of BQO is preserved under products) while Rado's order is not. Therefore one cannot characterise BQOs by the ordinal invariants considered here. Moreover, the two orders do not even embed into each other. To see this, assume by way of contradiction that f injects $\omega \times \omega$ into R . Write (a_i, b_i) and (c_i, d_i) for $f(0, i)$ and, resp., $f(i, 0)$ when $i \in \omega$. Necessarily the b_i 's and the d_i 's are unbounded. If the a_i 's are unbounded, one has the contradictory $f(1, 0) <_R f(0, i) = (a_i, b_i)$ for some i , and there is a similar contradiction if the c_i 's are unbounded, so assume the a_i 's and the c_i 's are bounded by some k . By the pigeonhole principle, we can find a pair $0 < i, j$ with $a_i = c_j$ so that $f(0, i) \not<_R f(j, 0)$, another contradiction. Hence $(\omega \times \omega) \not\leq R$. In the other direction $R \not\leq (\omega \times \omega)$, is obvious since $\omega \times \omega$ is BQO while R is not.

² We adopted the definition from Laver (1976).

Table 1 Ordinal invariants of the main WQOs.

P	$\mathbf{o}(P)$	$\mathbf{h}(P)$	$\mathbf{w}(P)$
$\alpha \in ON$	α	α	1 (or 0)
A_n (size n antichain)	n	1	n
Rado's R	ω^2	ω	ω
$\sum_{i \in \alpha} P_i$	$\sum_{i \in \alpha} \mathbf{o}(P_i)$	$\sum_{i \in \alpha} \mathbf{h}(P_i)$	$\sup_{i \in \alpha} \mathbf{w}(P_i)$
$P \sqcup Q$	$\mathbf{o}(P) \oplus \mathbf{o}(Q)$	$\max(\mathbf{h}(P), \mathbf{h}(Q))$	$\mathbf{w}(P) \oplus \mathbf{w}(Q)$
$P \cdot Q$	$\mathbf{o}(P) \cdot \mathbf{o}(Q)$	$\mathbf{h}(P) \cdot \mathbf{h}(Q)$	$\mathbf{w}(P) \odot \mathbf{w}(Q)$
$P \times Q$	$\mathbf{o}(P) \otimes \mathbf{o}(Q)$	$\sup_{\substack{\alpha < \mathbf{h}(P) \\ \beta < \mathbf{h}(Q)}} \alpha \oplus \beta + 1$	see Sec. 4.4
$M(P)$	$\omega^{\widehat{\mathbf{o}(P)}}$	$\mathbf{h}^*(P)$, see Sec. 4.5.2	see Thm. 4.27
$P < \omega$	$\omega^{\mathbf{o}(P) \pm 1}$, see Thm. 4.21	$\mathbf{h}^*(P)$	$\mathbf{o}(P < \omega)$
$T(P)$	$\vartheta(\Omega^\omega \cdot \mathbf{o}(P))$	$\mathbf{h}^*(P)$	$\mathbf{o}(T(P))$

5 Concluding Remarks

We provide in Table 1 a summary of our findings regarding ordinal invariants of WQOs. Mostly, the new results concern the width $\mathbf{w}(P)$ of WQOs. We note that the width $\mathbf{w}(P \times Q)$ of Cartesian products is far from elucidated, the first difficulty being that—unlike other constructs—it cannot be expressed as a function of the widths $\mathbf{w}(P)$ and $\mathbf{w}(Q)$. For Cartesian products, Sec. 4.4 only provide definite values for a few special cases: for the rest, one can only provide upper and lower bounds for the moment.

References

- Abraham U (1987) A note on Dilworth's theorem in the infinite case. *Order* 4(2):107–125, DOI 10.1007/BF00337691
- Abraham U, Bonnet R (1999) Hausdorff's theorem for posets that satisfy the finite antichain property. *Fund Math* 159(1):51–69
- Abraham U, Bonnet R, Cummings J, Džamonja M, Thompson K (2012) A scattering of orders. *Trans Amer Math Soc* 364(12):6259–6278, DOI 10.1090/S0002-9947-2012-05466-3
- Blass A, Gurevich Y (2008) Program termination and well partial orderings. *ACM Trans Comput Log* 9(3:18), DOI 10.1145/1352582.1352586
- Dilworth RP (1950) A decomposition theorem for partially ordered sets. *Ann Math* 51(1):161–166, DOI 10.2307/1969503
- Dushnik B, Miller EW (1941) Partially ordered sets. *Amer J Math* 63(3):600–610, DOI 10.2307/2371374
- Higman G (1952) Ordering by divisibility in abstract algebras. *Proc London Math Soc* (3) 2:326–336, DOI 10.1112/plms/s3-2.1.326
- de Jongh DHJ, Parikh R (1977) Well-partial orderings and hierarchies. *Nederl Akad Wetensch Proc Ser A* 80=Indag Math 39(3):195–207, DOI 10.1016/1385-7258(77)90067-1

- Kříž I, Thomas R (1990) Ordinal types in Ramsey theory and well-partial-ordering theory. In: Nešetřil J, Rödl V (eds) *Algorithms and Combinatorics vol. 5: Mathematics of Ramsey Theory*, Springer, pp 57–95, DOI 10.1007/978-3-642-72905-8_7
- Kruskal J (1960) Well-quasi-ordering, the Tree Theorem, and Vazsonyi's Conjecture. *Trans AMS* 95(2):210–225, DOI 10.2307/1993287
- Kruskal JB (1972) The theory of well-quasi-ordering: A frequently discovered concept. *J Comb Theory A* 13(3):297–305, DOI 10.1016/0097-3165(72)90063-5
- Laver R (1976) Well-quasi-orderings and sets of finite sequences. *Math Proc Cambridge* 79(1):1–10, DOI 10.1017/S030500410005204X
- Van der Meeren J (2015) *Connecting the two worlds: Well-partial-orders and ordinal notation systems*. PhD thesis, Universiteit Gent
- Van der Meeren J, Rathjen M, Weiermann A (2015) Well-partial-orderings and the big Veblen number. *Arch Math Logic* 54(1–2):193–230, DOI 10.1007/s00153-014-0408-5
- Milner EC (1985) Basic wqo- and bqo-theory. In: *Graphs and order (Banff, Alta., 1984)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol 147, Reidel, Dordrecht, pp 487–502, DOI 10.1007/978-94-009-5315-4_14
- Milner EC, Sauer N (1981) On chains and antichains in well founded partially ordered sets. *J London Math Soc* s2-24(1):15–33, DOI 10.1112/jlms/s2-24.1.15
- Mirsky L (1971) A dual of Dilworth's decomposition theorem. *Amer Math Month* 78(8):876–877, DOI 10.2307/2316481
- Peles MA (1963) On Dilworth's theorem in the infinite case. *Israel J Math* 1(2):108–109, DOI 10.1007/BF02759806
- Pouzet M (1979) Sur les chaînes d'un ensemble partiellement bien ordonné. *Publ Dép Math Lyon* 16(1):21–26, URL http://www.numdam.org/article/PDML_1979__16_1_21_0.pdf
- Rado R (1954) Partial well-ordering of sets of vectors. *Mathematika* 1(2):89–95, DOI 10.1112/S0025579300000565
- Rathjen M, Weiermann A (1993) Proof-theoretic investigations on Kruskal's theorem. *Ann Pure App Logic* 60(1):49–88, DOI 10.1016/0168-0072(93)90192-G
- Schmerl JH (2002) Obstacles to extending Mirsky's theorem. *Order* 19(2):209–211, DOI 10.1023/A:1016541101728
- Schmidt D (1979) *Well-partial orderings and their maximal order types*. Habilitationsschrift, Heidelberg
- Schmidt D (1981) The relation between the height of a well-founded partial ordering and the order types of its chains and antichains. *J Comb Theory B* 31(2):183–189, DOI 10.1016/S0095-8956(81)80023-8
- Schmitz S, Schnoebelen Ph (2011) Multiply-recursive upper bounds with Higman's lemma. In: *Proc. ICALP 2011*, Springer, Lect. Notes Comput. Sci., vol 6756, pp 441–452, DOI 10.1007/978-3-642-22012-8_35
- Weiermann A (2009) A computation of the maximal order type of the term ordering on finite multisets. In: *Proc. CiE 2009*, Springer, Lect. Notes Comput. Sci., vol 5635, pp 488–498, DOI 10.1007/978-3-642-03073-4_50
- West DB (1982) Extremal problems in partially ordered sets. In: *Rival I (ed) Ordered Sets*, NATO Advanced Study Institutes Series, vol 83, Springer, pp 473–521, DOI 10.1007/978-94-009-7798-3_16
- Wolk ES (1967) Partially well ordered sets and partial ordinals. *Fund Math* 60(2):175–186