# ITERATION OF STRONGLY $\kappa^+$-CC FORCING POSETS

JAMES CUMMINGS, MIRNA DŽAMONJA, AND ITAY NEEMAN

## 1. Introduction

One of the basic results in iterated forcing states that a finite support iteration of ccc forcing is ccc. It is natural to look for extensions of this result: the most natural setting for generalisations is to let $\kappa$ be an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$, and consider $< \kappa$-support iterations in which each iterand is $\kappa$-closed and $\kappa^+$-cc. It is known that (even for the case where $\kappa = \aleph_1$ and CH holds) such iterations do not in general have $\kappa^+$-cc [5], so we will need to strengthen the closure and chain condition hypotheses on the iterands.

Shelah [4] proved that if we strengthen the chain condition assumption a lot and the closure assumption a little then we get a useful iteration theorem. More precisely, let $\kappa = \kappa^{<\kappa}$ and say that a poset $\mathbb{P}$ is *regressively* $\kappa^+$*-cc* if it enjoys the following property: for every sequence $(p_i)_{i<\kappa^+}$ of conditions in $\mathbb{P}$ there exist a club set $E \subseteq \kappa^+$ and a regressive function $f$ on $E \cap \mathrm{Cof}(\kappa)$ such that $f(\alpha) = f(\beta)$ implies $p_\alpha$ is compatible with $p_\beta$. This looks technical, but can be motivated by the observation that if $\mathbb{P}$ was proved to be $\kappa^+$-cc by the standard $\Delta$-system and amalgamation arguments then the proof very likely shows that $\mathbb{P}$ is regressively $\kappa^+$-cc. Shelah's iteration theorem states that a $< \kappa$-support iteration with $\kappa$-closed, well met, and regressively $\kappa^+$-cc iterands is regressively $\kappa^+$-cc. Here a poset is *well met* if any pair of compatible conditions has a greatest lower bound (glb): Shelah [4] showed that in general this technical condition can not be removed.

We will prove an iteration theorem where the chain condition hypothesis is strengthened in a different direction. Motivation for this work includes some results by Mekler [2] where the ccc is proved using elementary submodels, and the more recent surge of interest (initiated by Mitchell's work on $I[\omega_2]$ [3]) in the notion of strong properness.

In Section 2 we give some background on forcing posets, elementary submodels and generic conditions. Section 3 contains the statement and proof of our main theorem. Finally Section 4 discusses some generalisations.

## 2. Background

For the rest of this paper we fix an uncountable regular cardinal such that $\kappa^{<\kappa} = \kappa$. We make the convention that when we write "$N \prec H_\theta$" we mean "$N \prec (H_\theta, \in, <_\theta)$" where $<_\theta$ is a wellordering of $H_\theta$. The structure $(H_\theta, \in, <_\theta)$ has definable Skolem functions, so that if $N, N' \prec H_\theta$ then $N \cap N' \prec H_\theta$. When $N \prec H_\theta$ we write $\bar{N}$ for the transitive collapse of $N$, $\rho_N : N \simeq \bar{N}$ for the transitive collapsing map, and $\pi_N : \bar{N} \simeq N$ for its inverse.

**Definition 1.** *Let $\mathbb{Q}$ be a forcing poset and let $M \prec H_\theta$. A model $M$ is $\kappa$-good for $\mathbb{Q}$ if and only if $\kappa, \mathbb{Q} \in M$, $|M| = \kappa$ and ${}^{<\kappa}M \subseteq M$.*

**Remark 1.** *If $\mathbb{Q} \in H_\theta$, then the set of $M$ which are $\kappa$-good for $\mathbb{Q}$ is stationary in $P_{\kappa^+} H_\theta$.*

When $M$ is $\kappa$-good for $\mathbb{Q}$ and $G$ is $\mathbb{Q}$-generic over $V$, we will study the subset $G \cap M$ of $\mathbb{Q} \cap M$. In a mild abuse of notation we sometimes write $\bar{G}$ for the subset $\rho_M[G \cap M]$ of the poset $\bar{\mathbb{Q}}$. We write $M[G]$ for the set of elements of form $\dot{\tau}^G$ where $\dot{\tau}$ is a $\mathbb{Q}$-name in $M$.

**Definition 2.** *Let $M$ be $\kappa$-good for $\mathbb{Q}$. Then:*

(1) *A condition $q \in \mathbb{Q}$ is* $(M, \mathbb{Q})$-generic *iff $q$ forces that $\bar{G}$ is $\bar{\mathbb{Q}}$-generic over $\bar{M}$, and* strongly $(M, \mathbb{Q})$-generic *iff it forces that $\bar{G}$ is $\bar{\mathbb{Q}}$-generic over $V$.*
(2) *If $q \in \mathbb{Q}$ and $r \in \mathbb{Q} \cap M$, then $r$ is a* strong properness residue of $q$ (for $M$) *iff for every $s \in \mathbb{Q} \cap M$ with $s \leq r$, $q$ is compatible with $s$. We write* spr *to abbreviate strong properness residue.*

Assume that $M$ is $\kappa$-good for $\mathbb{Q}$. The following facts are standard:

- If $G$ is $\mathbb{Q}$-generic over $V$, then $M[G] \prec H_\theta[G] = H_\theta^{V[G]}$. If in addition $\mathbb{Q}$ is $\kappa$-closed then $V[G] \models {}^{<\kappa}M[G] \subseteq M[G]$.
- A condition $q$ is $(M, \mathbb{Q})$-generic iff $q$ forces that $M[\dot{G}] \cap V = M$. In this case $q$ forces that $\pi_M$ can be lifted to an elementary embedding $\pi_M : \bar{M}[\bar{G}] \to H_\theta[G]$.
- A condition $q$ is strongly $(M, \mathbb{Q})$-generic iff the set of conditions in $\mathbb{Q}$ which have a spr for $M$ is dense below $q$.
- The poset $\mathbb{Q}$ is $\kappa^+$-cc iff every condition in $\mathbb{Q}$ is $(M, \mathbb{Q})$-generic.

**Definition 3.** *A forcing poset $\mathbb{Q}$ is* strongly $\kappa^+$-cc *if and only if for all large $\theta$, for every $M \prec H_\theta$ which is $\kappa$-good for $\mathbb{Q}$, every condition in $\mathbb{Q}$ is strongly $(M, \mathbb{Q})$-generic. Equivalently, densely many conditions have a spr for $M$, and this implies that in fact all conditions have a spr for $M$.*

## 3. An iteration theorem

**Theorem 1.** *Let $\kappa$ be uncountable with $\kappa^{<\kappa} = \kappa$. Let $\mathbb{P}$ be an iteration with $< \kappa$-supports such that each iterand $\mathbb{Q}_\alpha$ is forced at stage $\alpha$ to have the following properties:*

(1) *$\mathbb{Q}_\alpha$ is strongly $\kappa^+$-cc.*
(2) *$\mathbb{Q}_\alpha$ is well met.*
(3) *Every directed subset of $\mathbb{Q}_\alpha$ of size less than $\kappa$ has a glb.*

*Then $\mathbb{P}$ is strongly $\kappa^+$-cc.*

Depending on the exact way one defines "directed" in condition (3), condition (3) may be read to subsume condition (2).

Before proving the theorem, we digress briefly to illustrate the difficulties and motivate the main idea. Consider the case of an iteration $\mathbb{P}_2 = \mathbb{Q}_0 * \dot{\mathbb{Q}}_1$ of length two, where $\mathbb{Q}_0$ is strongly $\kappa^+$-cc and forces that $\dot{\mathbb{Q}}_1$ is strongly $\kappa^+$-cc. Let $M$ be $\kappa$-good for $\mathbb{P}_2$, and let $(q_0, \dot{q}_1)$ be an arbitrary condition for which we aim to construct a spr. If $r_0$ is a spr for $q_0$ and $M$, while $\dot{r}_1$ names a spr for $\dot{q}_1$ and $M[\dot{G}_0]$, then we are not warranted in claiming that $(r_0, \dot{r}_1)$ is a spr for $(q_0, \dot{q}_1)$. The issue is that while $\dot{r}_1$ names something which is the denotation of a term in $M$, there is no reason to think $\dot{r}_1$ itself is in $M$. In this simple case we can cope by first extending $q_0$ to some $q_0'$, which determines the identity of some term $\dot{r}_1'$ which denotes a spr for $\dot{q}_1$, and then choosing $r_0'$ which is a spr for $q_0'$: this clearly becomes problematic for an iteration of infinite length. We will deal with this kind of problem by building a spr on every relevant coordinate simultaneously. This is similar to the approach taken by [1], but without a need for side conditions.

**Remark 2.** *It is easy to see that condition 3 is preserved by iteration with $< \kappa$-supports, so that $\mathbb{P}$ satisfies it. To be explicit, if $D$ is a directed subset of $\mathbb{P}$ with $|D| < \kappa$ then we construct a glb $p$ for $D$ inductively. We build $p$ so that* $\operatorname{supp}(p) = \bigcup_{t \in D} \operatorname{supp}(t)$*: at stage $i$ we have that $p \restriction i$ is a glb for $\{t \restriction i : t \in D\}$, observe that $p \restriction i$ forces $\{t(i) : t \in D\}$ to be directed, and choose $p(i)$ to name a glb for this set.*

*Proof of Theorem 1.* Let the length of the iteration $\mathbb{P}$ be $\gamma$, let $\theta$ be sufficiently large and let $M$ be $\kappa$-good for $\mathbb{P}$. Let $p \in \mathbb{P}$ be arbitrary. We will produce $q \leq p$ such that $q$ has a spr $r$ for $M$.

We note that if $\alpha \in M \cap \gamma$ and $G_\alpha$ is $\mathbb{P}_\alpha$-generic, then it is routine to check that $M[G_\alpha]$ is $\kappa$-good for $\mathbb{Q}_\alpha$ in $V[G_\alpha]$. It follows that every condition in $\mathbb{Q}_\alpha$ has a spr for $M[G_\alpha]$.

We choose a certain auxiliary model $H$ such that $q, M \in H$ and $|H| < \kappa$. To construct $H$ we build an increasing chain of models $(H_i)_{i<\omega}$ and a strictly decreasing chain of conditions $(p_i)_{i<\omega}$ in $\mathbb{P}$ such that:

(1) For all $i$, $H_i \prec H_\theta$ and $|H_i| < \kappa$.
(2) $\operatorname{supp}(p) \cup \{p, M\} \subseteq H_0$.
(3) $p_0 = p$.
(4) For all $i$, $p_{i+1} \leq p_i$ and $p_{i+1} \in D$ for every dense open $D \in H_i$.
(5) For all $i$, $H_i \cup \operatorname{supp}(p_{i+1}) \cup \{p_{i+1}, H_i\} \subseteq H_{i+1}$.

We may choose $p_{i+1}$ because (using Remark 2) $\mathbb{P}$ is $\kappa$-closed. At the end we set $H = \bigcup_n H_n$. By Remark 2) the sequence $(p_n)$ has a glb $q$.

We record some information:

(1) By construction $H \prec H_\theta$, $|H| < \kappa$ and $p, M \in H$.
(2) By Remark 2, $\operatorname{supp}(q) = \bigcup_n \operatorname{supp}(p_n)$ and $q \restriction \alpha$ forces that $q(\alpha)$ is the glb of the sequence $(p_n(\alpha))$.
(3) If $g = \{x \in \mathbb{P} \cap H : \exists i\, p_i \leq x\}$, then $g$ is a filter on $\mathbb{P} \cap H$ which meets every dense open set in $H$.
(4) By definition, $q$ is the glb of $g$. We claim that $g = \{x \in \mathbb{P} \cap H : q \leq x\}$. Clearly if $x \in g$ then $q \leq x$, and if $x \notin g$ then by genericity there is $n$ such that $p_n \perp x$ and so $q \not\leq x$.
(5) We claim that the support of $q$ is $H \cap \gamma$. By construction $\operatorname{supp}(p_n) \subseteq H_n \cap \gamma$ for all $n$, and so $\operatorname{supp}(q) \subseteq H \cap \gamma$; conversely if $\alpha \in H \cap \gamma$ then by genericity there is $n$ such that $\alpha \in \operatorname{supp}(p_n)$.

The set $g \cap M$ is a directed subset of $\mathbb{P}$ and $|g \cap M| \leq |H| < \kappa$, so $g \cap M$ has an glb $r$. Since ${}^{<\kappa}M \subseteq M$, $g \cap M \in M$ and so by elementarity $r \in M$.

Main Claim: $r$ is a spr for the condition $q$ and the model $M$.

Proof of Main Claim: We let $s \leq r$ with $s \in M$ and build inductively a condition $q^*$ such that $q^*$ is a common refinement of $s$ and $q$. The induction is easy except at coordinates $\alpha \in \operatorname{supp}(s) \cap \operatorname{supp}(q)$, so fix such an $\alpha$. The support of $s$ is contained in $M$, and the support of $q$ is contained in $H$, so $\alpha \in H \cap M \cap \gamma$. Note that $s \leq r$ and by induction $q^* \restriction \alpha \leq s \restriction \alpha$, so that $q^* \restriction \alpha \Vdash s(\alpha) \leq r(\alpha)$.

For each $i < \omega$, define a set $D_i \subseteq \mathbb{P}$ as follows: $D_i$ is the set of $t \in \mathbb{P}$ such that either $t \perp p_i$, or $t \leq p_i$ and there is $\dot{r} \in M$ such that $t \restriction \alpha$ forces "$t(\alpha) \leq \dot{r}$, and

$\dot{r}$ is a spr for $p_i(\alpha)$ and $M[\dot{G}_\alpha]$". Since $\alpha, p_i, M \in H$ we have by elementarity that $D_i \in H$.

We claim that $D_i$ is dense. Let $t_0 \in \mathbb{P}$ be arbitrary. If $t_0$ is incompatible with $p_i$ then $t_0 \in D_i$, otherwise we find $t_1 \leq t_0, p_i$. Extending $t_1 \upharpoonright \alpha$ if necessary, we may assume that $t_1 \upharpoonright \alpha$ determines some $\dot{r} \in M$ which denotes a spr for $t_1(\alpha)$; now $t_1 \upharpoonright \alpha$ forces that $\dot{r}$ and $t_1(\alpha)$ are compatible so extending $t_1$ at coordinate $\alpha$ we obtain a condition $t_2 \leq t_1$ such that $t_2 \upharpoonright \alpha$ forces $t_2(\alpha) \leq \dot{r}$. Since $t_2 \leq t_1 \leq p_i$ we have $t_2 \upharpoonright \alpha \Vdash t_1(\alpha) \leq p_i(\alpha)$, so $t_2 \upharpoonright \alpha$ forces that $\dot{r}$ is a spr for $p_i(\alpha)$.

By the construction of the sequence $(p_i)$, we find $j$ such that $p_j \in D_i$. From the definitions $p_j \leq p_i$ (that is $j \geq i$), and $p_j \upharpoonright \alpha$ forces "$p_j(\alpha) \leq \dot{r}$ and $\dot{r}$ is a spr for $p_i(\alpha)$" for some $\dot{r} \in M$. As $p_j, p_i, \alpha, M \in H$ we may assume by elementarity that $\dot{r} \in M \cap H$. Now if we let $r^*$ be the condition in $\mathbb{P}$ that has $\dot{r}$ at coordinate $\alpha$ and is otherwise trivial, $p_j \leq r^* \in M \cap H$ so that $r^* \in g \cap M$.

So $r \leq r^*$, and since $q^* \upharpoonright \alpha \leq s \upharpoonright \alpha \leq r \upharpoonright \alpha$ we have $q^* \upharpoonright \alpha \Vdash r(\alpha) \leq r^*(\alpha) = \dot{r}$. Since also $q^* \upharpoonright \alpha \leq p_j \upharpoonright \alpha$, $q^* \upharpoonright \alpha$ forces that $\dot{r}$ is a spr for $p_i(\alpha)$. Since $q^* \upharpoonright \alpha \Vdash s(\alpha) \leq r(\alpha) \leq \dot{r}$, $q^* \upharpoonright \alpha$ forces that $s(\alpha)$ is compatible with $p_i(\alpha)$.

Now we force below $q^* \upharpoonright \alpha$ to obtain a generic object $G_\alpha$, and work in $V[G_\alpha]$ to compute a lower bound for the decreasing sequence $(s(\alpha) \wedge p_i(\alpha))$. Let $q^*(\alpha)$ name a lower bound, then $q^* \upharpoonright \alpha$ forces that $q^*(\alpha)$ is a lower bound for the sequence $(p_i(\alpha))$, and (since $q^* \upharpoonright \alpha \leq q \upharpoonright \alpha$) also that $q(\alpha)$ is the glb for the sequence $(p_i(\alpha))$. Hence $q^* \upharpoonright \alpha$ forces that $q^*(\alpha) \leq q(\alpha)$. Hence $q^* \upharpoonright \alpha \Vdash q^*(\alpha) \leq q(\alpha), s(\alpha)$ as required.

$\square$

## 4. Further results

With more work we can weaken the closure hypotheses on the iterands as follows: it is enough to assume that each iterand $\mathbb{Q}_\alpha$ is forced to be $< \kappa$-strategically closed, to be countably closed, and to satisfy the strengthened form of countable strategic closure in which move $\omega$ is required to be a glb for the moves played at finite stages.

The iteration theorem can also be generalised in other directions. For example let $S \subseteq \kappa^+ \cap \mathrm{Cof}(\kappa)$ be stationary, and define a poset to be $S$-strongly $\kappa^+$-cc if sprs exist for $\kappa$-good models $M$ with $M \cap \kappa^+ \in S$. Then $S$-strongly $\kappa^+$-cc forcing posets preserve the stationarity of $S$, and an iteration of $S$-strongly $\kappa^+$-cc posets with appropriate closure properties is $S$-strongly $\kappa^+$-cc. To prove the generalisation to $S$-strongly $\kappa^+$-cc posets, simply restrict throughout to $M$ such that $M \cap \kappa^+ \in S$.

We briefly sketch the proof of the generalisation weakening the closure hypothesis on the iterands.

We can construct $p_i$ and $q$ as in the proof of Theorem 1 from the weaker hypotheses. $p_{i+1}$ can be constructed using $< \kappa$-strategic closure. If $\sigma$ is a strategy for player II to produce descending chains of length $\omega$ with a glb, and taking $\sigma \in H_0$, one can use the fact that $p_{i+1}$ meets all dense open sets in $H_i$ to find a play $(u_n)_{n<\omega}$ by $\sigma$ so that $p_{i+1} \leq u_{2i+1} \leq u_{2i} \leq p_i$. This ensures that $(p_i)_{i<\omega}$ has a glb.

The final argument in the proof of Theorem 1, obtaining a lower bound for the sequence $(s(\alpha) \wedge p_i(\alpha))$, goes through with countable closure.

The only other use of closure in the proof is in defining $r$, a glb for $g \cap M$. We prove that this can be done with the weakened assumptions.

The support of $r$ is $M \cap H \cap \gamma$. We work by induction on $\alpha \in M \cap H \cap \gamma$ to define $r(\alpha)$, assuming that $r \upharpoonright \alpha$ has been defined and is a glb for $(g \cap M) \upharpoonright \alpha$. Passing to the transitive collapse $\bar{H}$ of $H$, we have that $\bar{g} = \rho_H[g]$ is generic over

$\bar{H}$ for $\rho_H(\mathbb{P})$. So $\bar{g} \restriction \alpha$ is generic for $\rho_H(\mathbb{P} \restriction \alpha)$ over $\bar{H}$, and $\bar{g}(\alpha)$ is generic for $\bar{\mathbb{Q}}_\alpha = \rho_H(\dot{\mathbb{Q}}_\alpha)[\bar{g} \restriction \alpha]$ over $H[\bar{g} \restriction \alpha]$.

By the strong chain condition, $\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \restriction \alpha]$ is generic over $\rho_H(M)[\bar{g} \restriction \alpha]$. Using the strategic closure of the $\alpha^{\text{th}}$ iterand it follows that for each $i$, there is a lower bound $w_i \in \bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \restriction \alpha]$ for $\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \restriction \alpha] \cap \rho_H(H_i)[\bar{g} \restriction \alpha]$. Let $\dot{\tau}_\alpha \in H \cap M$ be a strategy for player II to produce descending chains of length $\omega$ with a glb in $\dot{\mathbb{Q}}_\alpha$. Using the genericity of $\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \restriction \alpha]$ one can pick $w_i$ to be part of a play by $\rho_H(\dot{\tau}_\alpha)[\bar{g} \restriction \alpha]$. Let $\dot{w}_i$ name $w_i$. Note that by genericity the fact that the conditions $\dot{w}_i$ are part of a play by $\rho_H(\dot{\tau}_\alpha)$ is forced by conditions in $\bar{g}(\alpha) \cap \rho_H(M)[\bar{g} \restriction \alpha]$. Then $r \restriction \alpha$, being a lower bound for $(g \cap M) \restriction \alpha$, forces that the conditions $\pi_H(\dot{w}_i)$ are part of a play according to $\dot{\tau}_\alpha$, and therefore $(\pi_H(\dot{w}_i))_{i<\omega}$ has a glb. Let $\dot{r}(\alpha)$ name this glb. One can check that then $r \restriction \alpha + 1$ is a glb for $g \restriction \alpha + 1$.

## References


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Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh PA 15213-3890, USA

*E-mail address*: `jcumming@andrew.cmu.edu`

School of Mathematics, University of East Anglia, Norwich Research Park, Norwich NR4 7TJ, UK

*E-mail address*: `M.Dzamonja@uea.ac.uk`

Department of Mathematics, University of California, Los Angeles, Los Angeles CA 90095-1555, USA

*E-mail address*: `ineeman@math.ucla.edu`