Ramsey Methods and the problem DU

Mirna Džamonja*

Abstract

We consider Fremlin's notion of 1/2-density and the related notion of Fremlin cardinals. A well known related question is if every 1/2-dense hereditary family on an uncountable cardinal must have an infinite homogeneous family. These notions do not seem to lend themselves to Ramseyan methods. In particular, it is not known if a Fremlin cardinal must be a large cardinal. We introduce a related notion of 1/2-dense cardinals which is easier to handle using Ramsey methods. We show that a 1/2-dense cardinal must be at least strongly inaccessible. On the other hand, David Asperó showed that an ω-Erdös cardinal must be 1/2-dense. [1]

0 Preface

I was a Tutotial Speaker at the Young Set Theory Conference in Barcelona 2009. The topic of my lectures were Ramsey principles. I talked both about many successes of the applications of the Ramseyan methods in set theory, topology and analysis, and about one Ramsey-like problem that is still unsolved many years after it was posed. It is the problem of 1/2-density, which we explain below. Rather than writing an article about successes of Ramseyan methods, which are well documented in the literature (see for example [9], [2]), I have decided to explain in detail the problem of 1/2-density, bringing into it a Ramseyan perspective. There are several new results in this article but the answer to the main question 1.2 is still not known.

*School of Mathematics, University of East Anglia, Norwich NR4 7TJ, UK, m.dzamonja@uea.ac.uk

[1]My thanks go to EPSRC for their support through the grant EP/G068720 and to the organisers of the Young Set Theory 2009 in Barcelona for their invitation to give a tutorial.

1 Introduction

We start by some definitions.

Definition 1.1 *(i) A family $\mathcal{D}$ of finite subsets of a cardinal κ is* 1/2-dense *if for all finite $F \subseteq \kappa$ there is $F_0 \subseteq F$ with $F_0 \in \mathcal{D}$ and $|F_0| \geq 1/2 \cdot |F|$. $\mathcal{D}$ is* hereditary *if it is closed under subsets. Hereditary 1/2-dense families are called* 1/2-filling.

(ii) Suppose that $\mathcal{D}$ is a family of finite subsets of a cardinal κ and $H \subseteq \kappa$. Then H is homogeneous *for $\mathcal{D}$ if $[H]^{<\aleph_0} \subseteq \mathcal{D}$.*

The most interesting cardinals in the context of 1/2-dense families are ω_1 and $\mathfrak{c} = 2^{\aleph_0}$. The following questions appear as the problem DU on D.H. Fremlin's list (see [6]):

Question 1.2 *Suppose that $\mathcal{D}$ is a 1/2-filling family on ω_1.*

(i) (Argyros) Must there be an infinite set homogeneous for $\mathcal{D}$?

(ii) Under MA + ¬CH, *is it true that $\mathcal{D}$ must have an uncountable homogeneous set?*

It is known that under $\mathrm{cov}(\mathcal{N}) = \aleph_1$ in place of MA + ¬CH the answer to (ii) is negative, see [1] or [6] for the folklore proof. It is not known if the positive answer is consistent. A meaningful concept is obtained if Definition 1.1 is made with an arbitrary $\alpha \in (0,1)$ in place of 1/2, however it is known that this change does not add any generality. Namely, Fremlin [6] showed that the truth of the statement "every α-filling family $\mathcal{D}$ on κ has a homogeneous set of size λ" does not depend on $\alpha \in (0,1)$. Paper [4] gives a combinatorial characterisation of 1/2-filling families on ω_1 which have an uncountable homogeneous set under MA + ¬CH.

We use the notation $P(\kappa, \lambda)$ to state that every 1/2-filling family $\mathcal{D}$ on κ has a homogeneous set of size λ. This notation was introduced by Fremlin. We use the word 'homogeneous' and notation from the theory of partition relations to emphasise the intuition we expressed in [1], that $P(\kappa, \lambda)$ is a large cardinal statement. Along these lines, Fremlin proved in [6] that if κ is a real-valued measurable cardinal then $P(\kappa, \omega)$ holds, hence it is consistent modulo a measurable cardinal that $P(\mathfrak{c}, \omega)$ holds. On the other hand, it is an observation of Apter and Džamonja in [1], that if κ is λ-Erdös then $P(\kappa, \lambda)$

holds. The statement $\neg P(\mathfrak{c}, \omega)$ is of interest in analysis, as it can be used to construct interesting examples of spaces and functions. From this point of view, the former of the two large cardinal results is more interesting, as an Erdös cardinal is necessarily strongly inaccessible. On the other hand the consistency strength of Erdös cardinals is weaker than that of real valued measurable cardinals. Namely, the consistency strength of the existence of a real-valued measurable cardinal is that of a measurable cardinal, and the consistency strength of the existence of an Erdös cardinal is that the assumption that there $0^{\#}$ exists implies that in L there is an α-Erdös cardinal for every $\alpha < \omega_1$ (while from the existence of an ω_1-Erdös cardinal one can derive the existence of $0^{\#}$).

One difficulty in treating the problem has been that 1/2-density is a density notion which does not fit the classical treatment of partition relations. In this paper we explore the influence of Ramsey theory on this notion. For example, we show that there is a notion closely connected to 1/2-filling families and including 1/2-density which can be treated by classical Ramsey theory. Specifically, in §2 we show just in ZFC that there is a 1/2-dense family $\mathcal{D}$ of finite subsets of $\mathfrak{c}$ such that there is no infinite $X \subseteq \mathfrak{c}$ homogeneous for $\mathcal{D}$, and in fact that there is such a family on every cardinal below the first strongly inaccessible cardinal. The family is not hereditary. In view of Fremlin's result mentioned above, the result is optimal. However, as pointed out by M. Kojman, if we completely give up on the requirement of hereditariness, it is easy to give a trivial example of a 1/2-dense family $\mathcal{F}$ of any cardinal such that there is no infinite X homogeneous for $\mathcal{F}$, namely just taking the finite subsets with even cardinality will do. This family $\mathcal{F}$ has the property that there is no nonempty set homogeneous for $\mathcal{F}$. The family $\mathcal{D}$ constructed in §2 will have homogeneous sets of arbitrary finite size within every infinite set. In §3 we remark how these results relate to another known weakening of 1/2-fillingness.

In §4 we consider the problem of 1/2-density when restricted to sets of fixed finite size.

Following the notation from [1] we say that κ is a λ-*Fremlin cardinal* iff $P(\kappa, \lambda)$ holds, and when λ is ω we just speak of *Fremlin* cardinals. It is still not known if the first Fremlin cardinal must be a large cardinal. In §2 we introduce a related type of large cardinals, 1/2-dense cardinals, and we prove that such a cardinal must be strongly inaccessible. In a previous version of this article we asked if 1/2-dense cardinals exist. D. Asperó answered this by observing that in fact an ω-Erdös cardinal must be 1/2-dense. We give an argument for this at the end of §2.

2 1/2-dense cardinals

In this section, 1/2-dense cardinals will be defined as cardinals that satisfy a stronger version of Fremlin's property $P(\kappa,\omega)$. We have been interested in $P(\kappa,\omega)$ rather than $P(\kappa,\lambda)$ for $\lambda > \omega$, but many arguments in this section apply to $\lambda > \omega$ as well.

Definition 2.1 *Let $\kappa \geq \aleph_0$ be a cardinal. A family $\mathcal{D}$ of finite subsets of κ is said to satisfy $\varphi(\kappa)$ if the following properties hold:*

- *all singletons are in $\mathcal{D}$,*
- *$\mathcal{D}$ is a 1/2-dense family which has no infinite homogeneous set, and*
- *(spread property) for any infinite $A \subseteq \kappa$ there are subsets of A of arbitrarily large finite size which are homogeneous for $\mathcal{D}$.*

A cardinal κ such that $\varphi(\kappa)$ is not satisfied by any family of finite subsets of κ is said to be a 1/2-dense *cardinal.*

In other words, a cardinal κ is 1/2-dense if every 1/2-dense family of finite subsets of κ with the spread property and containing the singletons has an infinite homogeneous set. Clearly every 1/2-dense hereditary family has the spread property and contains the singletons, and therefore we have:

Observation 2.2 *A 1/2-dense cardinal is necessarily Fremlin.* ★

We may also observe that if a cardinal is 1/2-dense so are all the larger cardinals.

Lemma 2.3 *Suppose that λ is an infinite cardinal $\leq \kappa$ and there is a family $\mathcal{D}_\kappa$ satisfying $\varphi(\kappa)$. Then there is a family $\mathcal{D}_\lambda$ satisfying $\varphi(\lambda)$.*

Proof of the Lemma. Let $\mathcal{D}_\lambda = \mathcal{D}_\kappa \cap [\lambda]^{<\aleph_0}$. It is clear that $\mathcal{D}_\lambda$ is a 1/2-dense family of finite subsets of λ which has no infinite homogeneous set and which contains all singletons. Suppose that $A \subseteq \lambda$ is infinite, then there are subsets of A of arbitrarily large finite size which are homogeneous for $\mathcal{D}_\kappa$, and hence for $\mathcal{D}_\lambda$. $\bigstar_{2.3}$

We now prove that the first 1/2-dense cardinal is a large cardinal.

Theorem 2.4 *The first 1/2-dense cardinal, if it exists, is strongly inaccessible.*

Proof. Suppose that λ^* is the first 1/2-dense cardinal. By the example of Schreier family we know that $\lambda^* > \aleph_0$.[2]

Now we shall show that if $\kappa < \lambda^*$ then also $2^\kappa < \lambda^*$.

Lemma 2.5 *Suppose that there is a family $\mathcal{D}_\kappa$ satisfying $\varphi(\kappa)$. Then there is a family $\mathcal{D}_{2^\kappa}$ satisfying $\varphi(2^\kappa)$.*

Proof of the Lemma. Let $<^*$ be a fixed well-order of ${}^\kappa 2$ in order type 2^κ. We identify the cardinal 2^κ with the tree ${}^\kappa 2$ ordered by $<^*$. Let $K \subseteq {}^{\omega>}({}^\kappa 2)$ be the set of all $u = \langle x_0 <^* x_1 <^* \ldots <^* x_{r-1}\rangle$ where $r \geq 2$ is such that u is either $<_{\rm lex}$-increasing or $<_{\rm lex}$-decreasing.

If $x \neq y$ in ${}^\kappa 2$ we let $\Delta(x,y) = \min\{\alpha : x(\alpha) \neq y(\alpha)\}$. For u as above, we let $\delta(u) = \langle \Delta(x_0,x_1), \Delta(x_1,x_2), \ldots \Delta(x_{r-2},x_{r-1})\rangle$. Note that $\delta(u)$ is a finite sequence of ordinals $< \kappa$.

We let P_0 consist of all $u \in K$ such that $\delta(u)$ is strictly increasing. We define P_1 as the set of all those $u \in K$ for which $\delta(u)$ is strictly decreasing. Let $P = P_0 \cup P_1$.

Let $\mathcal{D} = \mathcal{D}_{2^\kappa}$ be given by

$$\mathcal{D} = \{\{f\} : f \in 2^\kappa\} \cup \{u \in P : \operatorname{ran}(\delta(u)) \in \mathcal{D}_\kappa\} \cup ([{}^\kappa 2]^{<\aleph_0} \setminus P).$$

Clearly $\mathcal{D}$ contains all singletons. To show that $\mathcal{D}$ is 1/2-dense in ${}^\kappa 2$ it suffices to consider $u \in P$. Let us first suppose that $u \in P_0$. If $r = |u| = 2$ then $|\delta(u)| \leq 1$ so $\operatorname{ran}(\delta(u)) \in \mathcal{D}_\kappa$. Otherwise, $\operatorname{ran}(\delta(u))$ is in any case a finite subset of κ and therefore there is $F \subseteq \operatorname{ran}(\delta(u))$ with $F \in \mathcal{D}_\kappa$ and $|F| \geq |\operatorname{ran}(\delta(u))|/2 = (|u|-1)/2$. F is the range of a sequence of the form $\langle \Delta(x_{i_0}, x_{i_0+1}), \ldots, \Delta(x_{i_k}, x_{i_k+1})\rangle$ for some $i_0 < i_1 < \ldots < i_k$ and $k \leq r-2$ and therefore F is not immediately seen to be of the form $\operatorname{ran}(\delta(v))$ for any $v \subseteq u$. However, since $u \in P_0$, we have that $\Delta(x_i, x_{i+1})$ increases with i. Therefore for every $s < k$ we have $\Delta(x_{i_s}, x_{i_{s+1}}) = \Delta(x_{i_s}, x_{i_s+1})$ and hence $F = \operatorname{ran}(\delta(v))$ for $v = \langle x_{i_0}, \ldots x_{i_{k+1}}\rangle$. Since $|v| = |F| + 1 \geq (|u|-1)/2 + 1 \geq |u|/2$, we have found $v \in \mathcal{D}$ as desired. The argument for $u \in P_1$ is similar.

[2]The Schreier family consists of finite subsets F of ω which satisfy $\min(F) \geq |F| + 1$, and the singleton $\{0\}$. This family is a well known example of a 1/2-dense hereditary family of subsets of ω for which there is no infinite homogeneous set.

Now suppose that X is infinite and homogeneous for $\mathcal{D}$, and assume simply that X has order type ω under $<^*$. Define a colouring c by colouring pairs $\{x, y\}$ in X colour 0 if the $<^*$ and $<_{\rm lex}$ order agree on $\{x, y\}$, and colour 1 otherwise. By Ramsey's theorem we can assume that all pairs are coloured the same colour and therefore X is $<_{\rm lex}$-increasing or $<_{\rm lex}$-decreasing. Suppose for simplicity that it is $<_{\rm lex}$-increasing, the argument in the other case is similar.

By induction on $n < \omega$ we choose $x_n \in X$ and a final segment B_n of X so that B_n are non-empty and decreasing, and $\Delta(x_n, x_m)$ for $n < m$ only depends on n. Let x_0 be the $<^*$-minimal element of X and $\xi_0 = \min\{\Delta(x_0, x_n) : n > 0\}$. Let $B_0 = \{x_n : \Delta(x_0, x_n) = \xi_0\}$, so clearly B_0 is non-empty. Suppose that $x_n \in B_0$ and $n < m$. Then $x_n <_{\rm lex} x_m$ by the assumptions above and $x_0 <_{\rm lex} x_n$. By the choice of ξ_0 we can only have $\Delta(x_0, x_m) \geq \xi_0$ and therefore it must be that $\Delta(x_0, x_m) = \xi_0$ and $m \in B_0$. Hence B_0 is a final segment of X. Now we let x_1 be the $<^*$-minimal element of B_0 and continue. Note that the sequence $\bar{\xi} = \langle \xi_n : n < \omega \rangle$ is strictly increasing.

At the end, by renaming, we can assume that $X = \{x_n : n < m\}$. Then note that $[X]^{<\aleph_0} \subseteq P_0$, exactly because $\bar{\xi}$ is strictly increasing. Hence for any $u \in [X]^{\geq 2}$ we have that $\mathrm{ran}(\delta(u)) \in \mathcal{D}_\kappa$. This means that $\{\Delta(x_n, x_{n+1}) : n < \omega\}$ is infinite homogeneous for $\mathcal{D}_\kappa$, a contradiction.

To show the final claim, suppose that A is an infinite subset of 2^κ, $2 \leq n < \omega$ and we shall find a $\mathcal{D}$-homogeneous subset of A of size $\geq n$. By an application of Ramsey's theorem we can assume as above that the order type of A under $<^*$ is ω, that $A = \{y_k : k < \omega\}$ is an $<^*$-increasing enumeration and that either A is $<_{\rm lex}$-increasing or $<_{\rm lex}$-decreasing. Note that since y_ks are binary sequences we must have that if $k_0 < k_1 < k_2$ then $\Delta(y_{k_0}, y_{k_1}) \neq \Delta(y_{k_1}, y_{k_2})$. Since there is no infinite decreasing sequence of ordinals we can thin A further if necessary to obtain that $\Delta(y_k, y_{k+1}) < \Delta(y_{k+1}, y_{k+2})$ for any k. Therefore, any finite subset of A gives rise to a sequence in P. Let $B = \{\Delta(y_k, y_{k+1}) : k < \omega\}$. By the inductive hypothesis, there is a subset of B of size n which is homogeneous for $\mathcal{D}_\kappa$. This implies as in the argument for 1/2-density that $\{y_k : \Delta(y_k, y_{k+1}) \in B\}$ is homogeneous for $\mathcal{D}_{2^\kappa}$. $\bigstar_{2.5}$

Our next task is to show that λ^* cannot be singular. For this we recast the property $\neg\varphi(\kappa)$ in terms of a classicaly-looking partition relation:

Definition 2.6 *We say that* $\kappa \rightarrow_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ *iff for every function* $f : [\kappa]^{<\omega} \rightarrow 2$ *satisfying* $f(\{\alpha\}) = 0$ *for all* α*,* *either* (i) *there is an infinite* $A \subseteq \kappa$ *such that* $f \restriction [A]^{<\omega}$ *is the constant* 0 *function,*

or (ii) *there is a finite* $B \subseteq \kappa$ *such that* $f \restriction [B]^{\lceil |B|/2 \rceil}$ *is the constant* 1 *function,*

or (iii) *there is an infinite* $A \subseteq \kappa$ *such that for some* $n < \omega$ *every* $B \in [A]^{\geq n}$ *has a subset* C *with* $f(C) = 1$.

Lemma 2.7 *A cardinal* κ *satisfies* $\neg\varphi(\kappa)$ *iff* $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ *holds.*

Proof of the Lemma. In the forward direction, given $f : [\kappa]^{<\omega} \to 2$ satisfying $f(\{\alpha\}) = 0$ for all α, define $\mathcal{D} = \{F : f(F) = 0\}$. If (ii) does not hold, then $\mathcal{D}$ is 1/2-dense. If (iii) does not hold then for every infinite $A \subseteq \kappa$ for every $n > \omega$ there is $B \subseteq A$ of size at least n whose all subsets are in $\mathcal{D}$. Since $\mathcal{D}$ cannot witness $\varphi(\kappa)$, there must be an infinite $\mathcal{D}$-homogeneous set, so (i) holds.

In the backward direction the proof is similar: if we are given a 1/2-dense family $\mathcal{D}$ of finite subsets of κ which contains all singletons and has the property that within every infinite subset of κ there is an arbitrarily large finite $\mathcal{D}$-homogeneous set, then we can define $f : [\kappa]^{<\omega} \to 2$ by $f(F) = 0$ iff $F \in \mathcal{D}$. Then 1/2-density of $\mathcal{D}$ implies that (ii) in $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ cannot hold and the property that within every infinite subset of κ there is an arbitrarily large finite $\mathcal{D}$-homogeneous set shows that (iii) cannot hold. Hence, (i) holds, and any infinite A witnessing it gives an infinite $\mathcal{D}$-homogeneous set. $\bigstar_{2.7}$

Lemma 2.8 λ^* *is not singular.*

Proof of the Lemma. By Lemma 2.7, this amounts to showing that the first κ satisfying $\kappa \to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$ cannot be singular. Suppose for contradiction that this is the case. Let $\kappa > \mathrm{cf}(\kappa)$ and let $\langle \kappa_i : i < \mathrm{cf}(\kappa) \rangle$ be an increasing continuous sequence of cardinals converging to κ, with $\kappa_0 = 0$ and $\kappa_1 \geq \omega$. For $\alpha < \kappa$ define $h(\alpha) = i$ iff $\alpha \in [\kappa_i, \kappa_{i+1})$. Let $f : [\mathrm{cf}(\kappa)]^{<\omega} \to 2$ exemplify that $\mathrm{cf}(\kappa) \not\to_0 (\omega, \lceil n/2 \rceil, \hat{\omega})^{<\omega}$, and let $f_i : [\kappa_{i+1}]^{<\omega} \to 2$ exemplify the same for κ_{i+1}.

Define $g : [\kappa]^{<\omega} \to 2$ as follows: for an increasing sequence $(\xi_1, \ldots, \xi_n)$ in $[\kappa]^{<\omega}$ let

$$g(\xi_1, \ldots, \xi_n) = \begin{cases} 0 & \text{if } n = 1, \\ f_{i+1}(\xi_1, \ldots, \xi_n) & \text{if } h(\xi_1) = \ldots = h(\xi_n) = i, \\ f(h(\xi_1), \ldots h(\xi_n)) & \text{if } h(\xi_1) < \ldots < h(\xi_n), \\ 0 & \text{otherwise.} \end{cases}$$

In our notation we use $g(\xi_1, \ldots, \xi_n)$ in place of $g(\{\xi_1, \ldots, \xi_n\})$, for clarity. We claim that g exemplifies that the required partition relation does not hold at κ. Clearly g maps all singletons to 0. Suppose that (i) holds, as shown by an infinite $A \subseteq \kappa$. Suppose that h"A is infinite. By thinning A if necessary we can assume that for $\xi < \zeta$ in A we have $h(\xi) < h(\zeta)$. Therefore $\{h(\zeta) : \zeta \in A\}$ gives an infinite subset of $\mathrm{cf}(\kappa)$ which shows that (i) holds for f, a contradiction. Otherwise h"A is finite and by thinning A if necessary we can assume that $h(\alpha)$ for $\alpha \in A$ is constant i. Then A shows that (i) holds for f_{i+1}, a contradiction. A similar contradiction is obtained assuming that (iii) holds for g. Finally suppose that $2 \leq n < \omega$ is given and $B \subseteq A$ has size n. If neither the second or the third clause of the definition of g apply to B, then $g(B) = 0$ and so B does not exemplify (ii). If either the second or the third clause applies to B, then so it does to any of its subsets, and hence there must be a subset C of B of size $\geq n/2$ which satisfies $g(C) = f_{i+1}(C) = 0$, or $g(C) = f(h^{-1}(C)) = 0$. So (ii) does not hold for g either, and hence we have a contradiction.$\bigstar_{2.8}$

We have now shown that λ^* must be strongly inaccessible, hence the theorem is proved. $\bigstar_{2.4}$

Remark 2.9 *To see specifically that $\mathcal{D}$ from the proof of Lemma 2.5 is not closed under subsets, say on 2^ω, notice for example that there are sequences in the complement of P with a subsequence in P_0 which is not in $\mathcal{D}$. The proof of Lemma 2.5 shows that all infinite subsets of ${}^\omega 2$ 'concentrate' on P_0 and that $\mathcal{D} \cap P_0$ is closed under subsets. However there is no uncountable subset of ${}^\omega 2$ all whose finite subsets are in P_0. To see this, suppose that A were such a set and let $\{x_\alpha : \alpha < \omega_1\}$ be a $<^*$-increasing subset of A. Let $n_\alpha = \Delta(x_\alpha, x_{\alpha+1})$. Then if $\alpha < \beta$ we have that $\{x_\alpha, x_{\alpha+1}, x_\beta, x_{\beta+1}\} \in P_0$, so $n_\alpha < \Delta(x_\alpha, x_\beta) < n_\beta$, letting us obtain an increasing ω_1-sequence in ω.*

Theorem 2.4 in conjunction with Fremlin's result that a real-valued measurable cardinal is Fremlin shows that 1/2-dense cardinals are strictly stronger than Fremlin cardinals. D. Asperó proved that any ω-Erdös cardinal is 1/2-dense, as we now show.

Theorem 2.10 (Asperó) *An Erdös cardinal is necessarily 1/2-dense.*

Proof. Recall that a cardinal κ is Erdös iff $\kappa \to (\omega)^{<\omega}$, which means that for every $f : [\kappa]^{<\omega} \to 2$ there is $H \in [\kappa]^{\omega}$ which is 'homogeneous' for f. In this context homogenous means that either there are unboundedly many $n < \omega$ such that $f''[H]^n = \{1\}$ or there is $n_0 < \omega$ such that for all $n \geq n_0$, we have $f''[H]^n = \{0\}$.

Suppose for contradiction that κ is an Erdös cardinal and that $\mathcal{D}$ is a 1/2-dense family of subsets of κ satisfying the property $\varphi(\kappa)$ from Definition 2.1. Let f be the following coloring of $[\kappa]^{<\aleph_0}$ into 2: $f(F) = 1$ iff all subsets of F are in $\mathcal{D}$. Let H be an infinite set homogeneous for f. By the spread property there cannot be $n_0 < \omega$ such that for all $n \geq n_0$, we have $f''[H]^n = \{0\}$. Therefore there are unboundedly many $n < \omega$ such that $f''[H]^n = \{1\}$.

Let $m < \omega$ be arbitrary and let $n \geq m$ be such that $f''[H]^n = \{1\}$. Therefore $[H]^{\leq n} \subseteq \mathcal{D}$ and hence $[H]^m \subseteq \mathcal{D}$. In conclusion, H is homogeneous for $\mathcal{D}$. $\bigstar_{2.10}$

3 Remarks on Fremlin cardinals

As mentioned before, Fremlin proved that a real-valued measurable cardinal must be, in our notation, a Fremlin cardinal. Modulo the existence of a measurable cardinal, it is consistent that $2^{\aleph_0}$ is a real-valued measurable cardinal, hence the analogue of Theorem 2.4 cannot be true for Fremlin cardinals. Remark 2.9 shows exactly where the proof would fail. However some of the techniques of the proof do apply. For example, we can easily prove the following theorem of Fremlin, the statement of which was communicated to us by Henryk Michalewski:

Theorem 3.1 *(Fremlin) There is a family of finite subsets $\mathcal{F}_{\mathfrak{c}}$ of $\mathfrak{c}$ such that $\mathcal{F}_{\mathfrak{c}}$ is closed under subsets, has no infinite homogeneous set, but for every $\alpha < \mathfrak{c}$ and $n < \omega$ there is $F \in \mathcal{F}_{\mathfrak{c}}$ with $|F| \geq n$ and $F \cap \alpha = \emptyset$.*

Proof. We use the notation of the proof of Theorem 2.4 with $\kappa = \omega$. Let $\mathcal{F}_\mathfrak{c}$ be the family of all sets $\{f_{\alpha_0}, \dots f_{\alpha_{n-1}}\}$ in ${}^\omega 2$ such that $\{\Delta(f_{\alpha_i}, f_{\alpha_j}) : i \neq j < n, \alpha_i < \alpha_j\}$ is in the Schreier family. This family of functions is clearly closed under subsets, and any infinite homogeneous set would give us an infinite homogeneous subset of the Schreier family. If $\alpha < \mathfrak{c}$ and $n < \omega$ is given, we can find a finite subset of $\{f_\beta : \beta > \alpha\}$ of the form $\{f_{\beta_0}, \dots, f_{\beta_{2n}}\}$ with β_i increasing with i and $\Delta(f_{\beta_i}, f_{\beta_{i+1}})$ also increasing with i. Then the set of such values has size $2n$ and it has a subset F of size n which is in the Schreier family. From F we can recover a subset H of $2n+1$ such that $\{f_{\beta_i} : i \in H\}$ satisfies $\{\Delta(f_{\alpha_i}, f_{\alpha_j}) : i < j, i, j \in H\} = F$ and then $\{f_{\beta_i} : i \in H\}$ is in $\mathcal{F}_\mathfrak{c}$. $\bigstar_{3.1}$

A version of this theorem was used by A. Avilés, G. Plebanek and J. Rodríguez in [3] to prove that there exists a weakly compactly generated Banach space X and a scalarly null function $f : [0,1] \to X$ which is not Mc Shane integrable. This answered several open questions in the theory of Mc Shane integration.

4 Homogenous sets of fixed exponent

It is natural to ask to what extent the problem of 1/2-density is linked to considering all finite subsets of a given set, rather than just finite sets of some bounded cardinality. We concentrate on ω_1 and observe that restricting to fixed cardinalities gives rise to infinite homogeneous sets of order type $\omega + 1$ for any 1/2-dense open family on ω_1:

Theorem 4.1 *Suppose that $n < \omega$ and $\mathcal{D}$ is a family of subsets of $[\omega_1]^{\leq n}$ closed under subsets and having the property that every element F of $[\omega_1]^{<\omega}$ has a subset F_0 of size at least $1/2 \cdot |F|$ such that $[F_0]^{\leq n} \subseteq \mathcal{D}$. Then there is a $A \subseteq \omega_1$ of order type $\omega + 1$ with $[A]^{\leq n} \subseteq \mathcal{D}$.*

If $n = 2$ then there is an uncountable such A, and in fact for any infinite κ if $\mathcal{D}$ is a family of subsets of $[\kappa]^{\leq 2}$ closed under subsets and having the property that every element F of $[\kappa]^{<\omega}$ has a subset F_0 of size at least $1/2 \cdot |F|$ such that $[F_0]^{\leq 2} \subseteq \mathcal{D}$, then there is $A \in [\kappa]^{\geq \omega}$ with $[A]^{\leq 2} \subseteq \mathcal{D}$.

Proof. Let $n < \omega$ be given. The following theorem seems to be folklore in partition calculus for ω_1: for all $n < \omega$,

$$\omega_1 \longrightarrow (\omega + 1, \omega + 1)^n.$$

If $\mathcal{D}$ is a family as in the assumptions, then we can define a colouring $f : [\omega_1]^n \to 2$ by letting $f(F) = 0$ iff $F \in \mathcal{D}$. The property of 1/2-density prevents any 1-homogeneous set of size $2n$, so there must be a 0-homogeneous set of order type $\omega + 1$.

Erdös-Dushnik-Miller theorem (see [5]) states that $\kappa \longrightarrow (\kappa, \omega)^2$ for any infinite κ, so the conclusion follows as in the previous argument. $\bigstar_{4.1}$

Corollary 4.2 *Suppose that $\mathcal{D}$ is a 1/2-dense open family on ω_1. Then for every $n < \omega$ there is a $A \subseteq \omega_1$ of order type $\omega + 1$ with $[A]^{\leq n} \subseteq \mathcal{D}$, and if $n = 2$ then there is an uncountable $A \subseteq \omega_1$ with $[A]^{\leq 2} \subseteq \mathcal{D}$.*

Proof. Suppose that $\mathcal{D}$ is a 1/2-dense open family on ω_1 and $n < \omega$. Let $\mathcal{D}_0 = \mathcal{D} \cap [\omega_1]^{\leq n}$. Then $\mathcal{D}_0$ satisfies the assumptions of Theorem 4.1, so the conclusions follow from the relevant parts of the Theorem. $\bigstar_{4.2}$

A natural way to build a 1/2-dense open family on ω_1 is to build for some fixed n a family $\mathcal{D}_0$ satisfying the assumptions of Theorem 4.1 and then to take $\mathcal{D} = \{F \in [\omega_1]^{<\omega} : [F]^{\leq n} \subseteq \mathcal{D}_0\}$. Corollary 4.2 says that such a family will always have a homogeneous subset of order type $\omega + 1$, and if $n = 2$, it will have an uncountable homogeneous subset.

We note that improvements are available for larger order types in the second coordinate of the Erdös-Dushnik-Miller theorem, for example if κ is regular then the original theorem has it as $\omega + 1$, and for κ singular one can consult [8] . For $n \geq 4$, it is well known in partition calculus that $(\omega_1) \not\rightarrow (\omega + 2, 5)^n$, as is $(\omega_1) \not\rightarrow (\omega + 2, \omega)^3$. Schipperus proved recently ([7]) that $\omega_1 \longrightarrow (\omega^2 + 1, 4)^3$.

Corollary 4.2 does not say anything about uncountable homogeneous sets with $n > 2$. The following is a well known folklore fact mentioned in the Introduction:

Fact 4.3 *Suppose that there are $\aleph_1$ many measure 0 sets whose union is $[0, 1]$ (i.e. $\mathrm{cov}(\mathcal{N}) = \aleph_1$). Then there is a 1/2-open dense family on ω_1 with no uncountable homogeneous sets.*

The proof (see [1] or [6]) uses compactness. In particular it also does not provide an answer to the question about uncountable homogeneous sets with $n > 2$ in the situation of Theorem 4.1. We state the question explicitly:

Question 4.4 *Suppose that $n \geq 3$ and $\mathcal{D}$ is a 1/2-dense open family on ω_1. Must there be a set $H \in [\omega_1]^{\aleph_1}$ with $[H]^n \subseteq \mathcal{D}$?*

References

[1] A. Apter and M. Džamonja, *Some remarks on a question of D.H. Fremlin regarding ε-density*, Archive for Mathematical Logic, vol. 40, no. 7, (2001) pg. 531-540.

[2] S. Argyros and S. Todorcevic, *Ramsey methods in analysis*, Advanced Courses in Mathematics. CRM Barcelona. Birkhuser Verlag, Basel, 2005.

[3] A. Avilés, G. Plebanek and J. Rodríguez, *The McShane Integral in Weakly Compactly Generated Spaces*, submitted (preprint version September 2009).

[4] M. Džamonja and G. Plebanek, *Generalisations of ε-density*, Acta Universitatis Carolinae Mathematica et Physica, vol. 44, no. 2, (2003), pg. 57-64.

[5] P. Erdös, A. Hajnal, A. Máté and R. Rado, *Combinatorial Set Theory: Partition Relations for Cardinals*, Akadémiai Kiadó, Budapest 1984.

[6] D.H. Fremlin, *Problem DU*, version 3.9.05, available on the web site http://www.essex.ac.uk/maths/staff/fremlin/problems.htm

[7] R. Schipperus, *On the Milner-Prikry Relation*, preprint 2006.

[8] S. Shelah and L. Stanley, *Filters, Cohen sets and consistent extensions of the Erdös-Dushnik-Miller theorem*, Journal of Symbolic Logic, vol. 65, no. 1, (2000), pg. 259-271.

[9] S. Todorčević, *Partition problems in topology*, Contemporary Mathematics, vol. 84. American Mathematical Society, Providence, RI, 1989.