András Hajnal, life and work

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This July 30, 2016, the international set theory community lost one of its greatest, long standing, contributors: András Hajnal. He was known for his many theorems, including the Hajnal Free Set Theorem, partition calculus, where together with Erdős and Rado he was a founding father, and the theory of set mappings. He is also, with Galvin, the author of a celebrated theorem in cardinal arithmetics which was a precursor to Shelah's pcf theory. Although mostly known for his work on combinatorial set theory, Hajnal contribued to the study of constructibility, in an early work that extended the work of Gödel by introducing the idea of relative constructibility. He also made major contributions to finite combinatorics, including his theorem with Szemeredi on equitable coloring of graphs that proved a conjecture of Erdős. In this article we shall briefly speak about Hajnal's life and then review some of his greatest theorems.

András Hajnal's life

András Hajnal was born on May 13, 1931 in Budapest, the city where he spent many years, from which he moved to the United States, to which he eventually returned at the end of his life, and where he died. A large part of his career is closely connected to the Eötvös Loránd University (ELU) in Budapest, where he received his university diploma in 1953, his Candidate of Mathematical Sciences degree in 1957 (under the supervision of László Kalmár) and his Doctor of Mathematical Science degree in 1962. From 1956 to 1995 he was a faculty member at ELU. However, in 1994 he moved to Rutgers University (USA) to become the director of the centre DIMACS, and he remained there as a professor until his retirement in 2004.

Hajnal was an Honorary President of the European Set Theory Society and, since 1982, a member of the Hungarian Academy of Sciences, where he directed its mathematical institute from 1982 to 1992. He was the general secretary of the János Bolyai Mathematical Society from 1980 to 1990, and president of the society from 1990 to 1994. Since 1981, he has been an advisory editor of the journal Combinatorica. In 1992, Hajnal was awarded the Officer's Cross of the Order of the Republic of Hungary. Hajnal's influence on mathematics and mathematicians is enormous. We shall discuss Hajnal's mathematics shortly, as for mathematicians, some of his students were Miklos Ajtai, Richard Carr, Peter Hamburger, Istvan Juhász, Peter Komjáth, György Petruska and Lajos Soukup. Some of his mathematical grandchildren are Marianna Csörnyei and Miklós Laczkovich. Generations of mathematicians in Hungary learned their set theory from a book written by Hajnal and Mate, which is since 1999 available in English, in an updated version authored by Hajnal, Mate and Hamburger. My students in England and in France invariably get this book on their reading list and they love it.

The writer of these lines did not have the chance to co-author a paper with Hajnal, but we were good friends and colleagues. I saw him regularly at Rutgers, where I was a frequent visitor and I had the honour to speak at the 1999 MAMLS conference at Rugers which was devoted to Hajnal. We were sad at that conference since it was known that he had been diagnosed with lung cancer. But, in spite of the odds, he made it! He fully recovered and a happy '80th birthday of Hajnal' conference was held at Rutgers in 2011. In 2007 my mother was diagnosed with lung cancer and I asked András for advice and contacts. He generously shared all he knew, but unfortnately it did not work neither for my Mum, who died in 2009, nor for Hajnal's own wife Emilia who died of lung cancer in 2015. I wrote to him 'You had such a wonderful life together and were an example of a couple whose love lasted a lifetime.' I have heard that András was no longer the same after she died, and he left us too, suddenly, of a heart attack. The last correspondance we had dates from 1st of July 2016, just a few weeks before his death, when we discussed the compactness of the chromatic number of graphs. Although he claimed the he was slower than before, I found him totally up to it and I am sorry that we could not continue these discussions later.

Emilia and András had one son, Peter, who is a very successful scientist in his own right.

Hajnal's Mathematics

All together, Hajnal published over 155 papers and four books. One of the books is a celebrated bible of the partition calculus, "Combinatorial set theory. Partition relations for cardinals", co-authored with Erdős, Maté and Rado. Another one is the book "Set Theory" which we mentioned above (in Hungarian and in English) and, in addition, he wrote a school manual on graph theory for school children. He also edited 7 volumes of mathematical papers. Hajnal's papers were written in three different languages: Hungarian, German and English.
A complete list of Hajnal's papers is available on his web page at
https://www.renyi.hu/~ahajnal/hajnalpu.pdf
We refer to this publication list for references.

The first association that Hajnal's name gives us is the combinatorial set theory, including his many papers with Paul Erdős. But his first work was on something entirely different: in his Ph.D. thesis in 1961 (but already announced in 1956) he introduced the models $L(A)$ and proved that if κ is a regular cardinal and A is a subset of κ^+, then ZFC and $2^\kappa = \kappa^+$ hold in $L(A)$. This can be applied to prove relative consistency results: e.g., if $2^{\aleph_0} = \aleph_2$ is consistent then so is $2^{\aleph_0} = \aleph_2$ and $2^{\aleph_1} = \aleph_2$. This was before the invention of forcing which gave a tool for proving such consistency results.

Moving on, we quickly arrive, in 1961 (paper 12 on his list of publications), to the celebrated Hajnal's free set theorem. Suppose that we have a set S of size κ, a cardinal λ and a function $f:S\rightarrow[S]^{<\lambda}$. A subset S' of S is *free* if for every X,Y in S' we have that X does not belong to f(Y) and vice versa. Ruziewicz had conjectured that in this situation there must be a free set of size λ. Continuing a line of partial results by eminent authors, Hajnal finally confirmed this conjecture in a "surprisingly simple and ingenious way", as said Paul Erdős.

Hajnal is famous for his work on partition relations for cardinals, much of it in collaboration with Paul Erdős. Indeed, together they published 56 papers, both in finite and in infinite combinatorics. They also largely influenced the international community by publishing papers containing a list of open questions. He was also a majr contributor in the partition calculus of ordinals, including his result with Baumgartner (1973) that for every partition of pairs of vertices of the complete graph on $\aleph_1$ vertices into finitely many subsets, at least one of the subsets contains a complete graph on α vertices for every countable α. This result contains a new idea in the method of proof, since it was first proved under MA and then converted into a ZFC result by absoluteness.

With Juhàsz (now a corresponding member of the Hungarian Academy of Sciences), Hajnal worked on set-theoretic topology and they were the first ones (1968) to construct an S-space and an L-space.

In graph theory, Hajnal made contributions both in the finite and the infinite domains. A celebrated construction is his construction of two graphs of chromatic number $\aleph_1$ whose product is countably chromatic (1985). This shows that Hiedetniami conjecture is false for the infinite. In finite graph theory, probably his most well known result is The Hajnal–Szemerédi theorem (1970) on equitable coloring, proving a 1964 conjecture of Erdős: let Δ denote the maximum degree of a vertex in a finite graph *G*. Then *G* can be colored with $\Delta + 1$ colors in such a way that the sizes of the color classes differ by at most one. Hajnal has several important papers in graph theory with his former student Komjáth, a member of Hungarian Academy in his own right.

In a different part of set theory, Hajnal proved together with Galvin (1975, Annals of Mathematics) a result that was very unexpected at the time: if $\aleph\omega_1$ is a strong limit cardinal then 2^{$\aleph_{\{\omega_1\}}$}<ℵ_{$2^{\{\aleph_1\}^+}$} .
This was the result that initiated Shelah's pcf theory.

Hajnal had many other great contributions and continued producing mathematics to the very end of his life.

What else to say? All great men die but behind some of them, their theorems remain. Hajnal was in this class.