Interview with a set theorist

Mirna Džamonja and Deborah Kant

Abstract The status of the set-theoretic independent statements is the main problem in the philosophy of set theory. We address this problem by presenting the perspective of a practising set theorist, and thus giving an authentic insight in the current state of thinking in set-theoretic practice, which is to a large extent determined by independence results. During several meetings, the second author has been asking the first author about the development of forcing, the use of new axioms and set-theoretic intuition on independence. Parts of these conversations are directly presented in this article. They are supplemented by important mathematical results as well as discussion sections. Finally, we present three hypotheses about set-theoretic practice: First that most set theorists were surprised by the introduction of the forcing method, second that most set theorists think that forcing is a natural part of contemporary set theory, and third that most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength, and for most set theorists, a difference in consistency strength weighs much more than the difference between forcing axiom and large cardinal axiom.

Introduction

The current situation in set theory is an exciting one. In the 1960s, set theory was challenged by the introduction of the forcing technique, in reaction to which some researchers might have turned their back on set theory, because it gave rise to a vast range of independence results. However, today, the independence results constitute

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a large part of set-theoretic research. But how do set theorists think about it? How do mathematicians think about provably independent statements?

An answer to this question can be attempted through a detailed description of the current set-theoretic practice through the eyes of set-theorists themselves, which is an enterprise to be realised. The present article provides a step towards that program by giving a description of some important aspects of set-theoretic practice, formulated and observed from a joint mathematical and philosophical perspective. During several meetings in Paris[1], the second author (PhD candidate in philosophy of set theory) has been talking to the first author (a logician specialising in set theory and a professor of mathematics) in order to gain insights into the current situation in set theory, and to understand how set theorists think about their work and their subject matter. Parts of these conversations are directly presented in this article. They are supplemented by descriptive paragraphs of related (mathematical) facts as well as comments and discussion sections.

The article is structured as follows. At first, we argue for the relevance of this article and our method. The second section contains facts of set theory and logic that will be relevant in the following sections. In the third section, we present some important forcing results, which includes mathematical details but it is self-contained, and we added many (historical and mathematical) references. We then elaborate on classical, philosophical thoughts that can be found in set-theoretic practice, for instance the idea of platonism. The fifth section presents some general observations about independence: Set theorists have developed a very good intuition which problems might turn out independent, and which ones might be solvable in ZFC, and they can organise and differ between set-theoretic areas in this respect. Finally, we present three hypotheses about set-theoretic practice: First that most set theorists were surprised by the introduction of the forcing method, second that most set theorists think that forcing is a natural part of contemporary set theory, and third that most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength, and for most set theorists, a difference in consistency strength weighs much more than the difference between forcing axiom and large cardinal axiom.

The intended audience of this article are set theorists on the one hand and philosophers of mathematics on the other. Set theorists can skip section 2 where important mathematical concepts are introduced. For philosophers of mathematics who do not focus on set theory in their work, this section is intended to prepare an understanding of the following text. In that case, section 3 can be scanned without a loss of understanding for the subsequent sections. The aim of section 3 is to show a variety of applications of the forcing method, and to support the view that forcing is an integral part of set-theoretic research.

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1 Methodological background

We are interested in describing and analysing what set theorists are doing. We adopt the concern that Rouse describes: “A central concern of both the philosophy and the sociology of science [is] to make sense of the various activities that constitute scientific inquiry”[2], and apply it to set theory with a special focus on independence. In order to take a first step in the right direction, we present one specific view in current set-theoretic practice: the view of the first author on her discipline. We cannot generalise this view to a description of set-theoretic practice, because other set theorists might have other views. Thus, we see the present article as an attempt to start a discussion on set-theoretic practice, which should lead to a more general, rigorous analysis.

How to describe and analyse set-theoretic practice?

This question certainly deserves much more attention than we can devote to it in this article.[3] We briefly state our main points.

Our methods consist of a sociological method and philosophical thinking. On the sociological side, we can choose between two main methods, which are surveys and interviews. Surveys have the advantage of an easy evaluation of the information. They are suitable to check clearly formulated hypotheses. But there are two problems concerning hypotheses on set-theoretic practice: A missing theory and the question of a suitable language.

The study of set-theoretic practice is a very new research area. In the literature, one can find specific views about set-theoretic practice,[4] or investigations of specific parts of set-theoretic practice,[5] but there is no analysis of current set-theoretic practice in general. This means, that we do not have a theory at hand, which could be tested in a survey. But we can use interviews to find reasonable hypotheses about set-theoretic practice.

Secondly, not only the theory has to be developed but also a suitable language for the communication between set theorists and philosophers has to be found. There are sometimes huge differences between the language which philosophers use and the language which set theorists use to talk and think about set theory. Therefore, communication between both disciplines can be difficult. There is a greater risk of a misunderstanding in a survey already formulated by a philosopher and later answered by a mathematician than in a question asked in an interview in which the possibility to clear misunderstandings is directly given. Hence, based on these aims of finding reasonable hypotheses and of a successful communication, we decided to do interviews.

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[2] (**?**, p.2)

[3] But it will be considered more attentively in the PhD project of the second author.

[4] **?** and **?**

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On the philosophical side, a mathematical perspective is brought together with philosophical ideas on mathematics. Furthermore, a mathematical perspective is transferred to philosophy, i.e. presented in a way that makes it comprehensible to other philosophers. The elaboration on our method as it is happening in this section is part of the philosophical side of the methodology.

Why describe and analyse set-theoretic practice?

We argue here that set-theoretic knowledge is not completely captured by gathering together all theorems, lemmas, definitions etc. and the mathematical motivations and explications that mathematicians give to present their research. For example, when set theorists agree that cardinal invariants are mostly independent to each other, then such judgements, that are based on experience, are argued to be part of set-theoretic knowledge as well. Such judgements can be found in set-theoretic practice. Hence, we want to extend our focus to set-theoretic practice in general. The notion of set-theoretic practice is taken here very generously as including all mathematical activities performed by set theorists, and their thoughts and beliefs about mathematics (where the latter also includes definitions and theorems because we assume that set theorists believe what is an established definition and what was proven). In order to learn more about set-theoretic knowledge, we think that it is valuable to present the practices of the discipline, the similarities and differences between the views of set theorists, and to formulate general ideas about the current situation in set theory.

The reasons for this are at least three, as we describe.

Visions Reflecting on the practices, the historical developments, the importance or role of specific objects and methods etc. gives rise to formulated visions for future set-theoretic research. What would set theorists wish to find out in, let us say, the next ten years?[6]

Availability Set-theoretic knowledge is not easily accessible to other mathematical areas, the Sciences in general and the philosophy of mathematics. With important exceptions, such as the philosopher Alain Badiou[7] or the musician François Nicolas[8] and of course many mathematicians and philosophers of mathematics, the independence phenomenon is not sufficiently known outside of the community directly studying set theory and logic. In this respect, the formulation of set-theoretic ideas in a general and simple language could make set-theoretic knowledge available to other researchers who are interested in set theory.

Reasonable premises in philosophy In the philosophy of set theory, there is a debate on the new axioms and the independence problem. Typical questions are

[6] Description of large research programs is already explicitly described by some set theorists. Consider, for example, the research programs by Hugh W. Woodin or Sy-David Friedman. For Woodin's program *see* the large description of the current research state **?**, and for the S.D. Friedman's Hyperuniverse Program, *see* for example a founding article **?**. Not everybody describes their program so specifically and we wish to discover more about these unspecified programs.

[7] **?** bases ontology on the set-theoretic axioms and also considers forcing.

[8] www.entretemps.asso.fr/Nicolas/

Is every set-theoretic statement true or false? or *Which criteria can justify the acceptance of a new axiom?*[9] In some important philosophical approaches, set-theoretic practice plays a major role. For example, in Maddy's approach, philosophical questions can only be answered when considering in detail what set theorists are doing.[10] Hamkins' multiverse view is also strongly motivated by the current situation in set-theoretic research.[11] Therefore, some existing philosophical ideas are to be complemented by an analysis of set-theoretic practice,[12] and such an analysis seems in general a promising starting point for future philosophical research.

2 Preliminary facts

We summarise basic facts of logic and set theory such as the twofold use of the concept of set, the incompleteness of an axiomatic theory, independence proofs, the continuum hypothesis, forcing, and the new axioms. This section is intended to prepare the understanding of the following interview parts, and it contains the necessary background for philosophical questions on set theory.

Set theory is the study of sets, and sets are determined by their elements. We can take the union of two sets, we can take their intersection, we can build ordered pairs and sequences. And we can consider infinite sets, like the set of all natural numbers, or the set of all real numbers. Numbers can themselves be interpreted as certain sets;[13] functions as well, and many more mathematical objects. But also a formal sentence can be interpreted as a certain set. Every symbol of the formal language is interpreted as a set and then a sentence is just a finite sequence of these sets.[14] Therefore, also a formal theory—a set of axioms and all formal sentences that can be derived from these axioms—can be interpreted as a certain set. In set theory, the standardly used, formal theory is the Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC).

Thus, one could say that the concept of set in mathematics and in logic has an extremely wide scope of application. As a consequence, set theorists study sets, and at the same time, they study the different models of ZFC (starting from the assumption that there is a model of ZFC); they can use their own sophisticated, set-theoretic methods on the models. (By contrast, when considering models of number theory we cannot use number theory itself to manipulate the models, one would have

9 *See* the article by Laura Fontanella in this voume.

10 *See* for example **?** and **?**.

11 **?**

12 Thanks to Carolin Antos for emphasising this fact.

13 $0=\emptyset, n+1=n\cup\{n\}, \mathbb{N}=\{n : n<\omega\}, \mathbb{R}=\mathscr{P}(\mathbb{N})$, and so on. (These equations should not be understood as the claim that numbers *are* sets.)

14 For example, take the coding '$\exists$'=8, '$\forall$'=9, '$\neg$'=5, '('=0, ')'=1, 'v_0'=(2,0), 'v_1'=(2,1), then the statement '$\exists v_0 \forall v_1 (\neg v_1 \in v_0)$' can be coded as the sequence $\langle 8,(2,0),9,(2,1),0,5,(2,1),4,(2,0),1\rangle$.

to use set theory.) Because of this twofold application of the concept of set, these two levels, which are sometimes distinguished as mathematics and metamathematics, are closely intertwined in today's set-theoretic practice.

The study of different models of ZFC is also the study of the independence phenomenon in set theory. The starting assumption to study a model is always that there exists a model of ZFC (by Gödel's Completeness Theorem for the first order logic, this is equivalent to the assumption that ZFC is consistent).[15] This assumption itself is not part of ZFC (it could not be because of Gödel's second Incompleteness Theorem), but it is a natural assumption in practice—what should be retained here is that the set-theoretic method of building models is not a constructive method because of this starting assumption. If then, for a sentence φ, set theorists can build a model of the theory ZFC$+\varphi$ and they can also build a model of the theory ZFC$+\neg\varphi$, then φ is an independent sentence.

Let us give one example. In set theory, two sets have the same size if there is a one-to-one onto correspondence between them. So, the set of the natural numbers has the same size as the set of the even, natural numbers, because $f : n \mapsto 2n$ is a one-to-one onto correspondence between them. All members of the first set can be completely paired up with the members of the second set. But if we take the real numbers as the second set, they cannot be completely paired up with the natural numbers (this is Cantor's Theorem). This gives rise to different sizes of infinite sets.

The size of sets is measured by cardinal numbers. 0, 1, 2, ... are cardinal numbers. For example, the empty set, $\emptyset$, has size 0, and the set that contains as its only element the set of natural numbers, $\{\mathbb{N}\}$, has size 1. The set of the natural numbers itself, $\mathbb{N}$, has size $\aleph_0$, which is the first infinite cardinal number (set theorists always start counting at 0). Of course, there are further cardinal numbers: $\aleph_1$, $\aleph_2$, ... Now, set theorists have built models in which there are exactly $\aleph_1$ real numbers, and they have built models in which there are exactly $\aleph_2$ real numbers. And thus, the sentence "there are exactly $\aleph_1$ real numbers" (the famous Continuum Hypothesis (CH)) is an independent sentence. This can only be shown by building such models, and the most powerful technique to build such models is forcing.

Forcing was introduced in 1963 by Paul Cohen,[16] who showed by this method that the Continuum Hypothesis is independent. The method was then adopted by the set theorists who found since then (and continue to find) many independent statements. Different problems require different variations of the forcing method so that many kinds of forcing have been developed. This led to the formulation of forcing axioms. Such an axiom can be added to ZFC in order to facilitate the application of forcing. A forcing axiom for a certain kind of forcing states that any object that can be forced to exist by that kind of forcing, already exists; it states that the forcing method already has been applied. These axioms are part of the new axioms in set theory.

It should be noted that the notion of a new axiom is rarely used by set theorists. But in the philosophy of set theory, this notion includes all the axioms which are not

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[15] Of course, any stronger assumption works as well, in particular any Large Cardinal Axiom.

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part of the standard axiomatisation ZFC, but which are considered in set-theoretic practice.

In addition to the Forcing Axioms, there is another important class of new axioms—the Large Cardinal Axioms, which state that there exists a certain large cardinal. The smallest known large cardinal is an inaccessible cardinal. Other important large cardinals are measurable cardinals, Woodin cardinals, and supercompact cardinals (ordered by increasing strength). The existence of such large cardinals is not provable in ZFC (but for all we know, it might be that ZFC proves the non-existence of some of them!).

There are further statements, for instance determinacy statements, which are sometimes considered as new axioms, e.g. the Axiom of Determinacy (AD) which is consistent with ZF but contradicts the Axiom of Choice, and Projective Determinacy (PD) which is implied by the existence of infinitely many Woodin cardinals (Martin-SteelTheorem, 1985).

3 Some Important Forcing Results

This section presents briefly important steps in the development of the forcing technique. We first describe the first authors' perspective on the moment of the introduction of forcing and the process of its adoption by set theorists. Second, we give an overview on subsequent inventions of different kinds of forcing, the conjectures they solved, and the formulation of Forcing Axioms.[17]

Cohen's Introduction of Forcing

Before the introduction of forcing by Paul Cohen in 1963[18], there were no substantial independence results in set theory. Concerning the most famous independent sentence, the Continuum Hypothesis, it was already known that it cannot be refuted, because Gödel gave a model, L, in which it holds. L is an inner model of ZFC which is not obtained by the forcing method. M. Džamonja thinks that "many practising set theorists at that time were hoping or were assuming that CH or GCH[19] would be proven to be true." And she refers to a similar situation today: "Maybe just like now, we think that Large Cardinal Axioms are true in a sense, even though we cannot prove that they are." Today, Large Cardinal Axioms are an integral part of set-theoretic research. In comparison to other new axioms, they are the most acceptable ones. In general, most set theorists trust in the consistency of these axioms and do not believe that assuming them causes any harm. In other words, if

[17] Readers who are further interested in the mathematical details of forcing are referred to **?** for an introduction, or **?** for a classical presentation of forcing, or **?** for a presentation of the forcing methods that are used today.

[18] *See* **?** and **?**.

[19] General Continuum Hypothesis: For every ordinal α, it holds $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$.

one would have to choose between the Large Cardinal Axioms and their negations, most set theorists would not choose the negations. Following this analogy, imagine now that this was the case with the Continuum Hypothesis before Cohen's result, which would mean that most set theorists did not expect that the negation of CH is a reasonable statement to consider[20] This situation explains why Cohen's proof could have been such a surprising result. In the following dialogue, we are speaking about this moment, how the subsequent adoption of forcing by the set theorists proceeded, and in particular how forcing turned from a newly introduced tool into a natural part of set theory.

D. Kant: *When Cohen's result was published, was it regarded as unnatural?*

M. Džamonja: *It was regarded as something very unnatural. There were many people who stopped working in set theory when they found out about this result. One of them was P. Erdös. He was a most prominent set theorist who had proved many interesting results but he just didn't think that forcing was an interesting method or that it brings anything. Well, he has this famous statement that 'independence has raised his ugly head'. So, he didn't like it, he never learned the method. And I think, in general, it was regarded of course as a big surprise. Cohen got a Fields medal for it. But it was very esoteric and I think that even Cohen himself, did not understand it the same way that we understand it now, after so many years, of course. Things become easier after many people had looked at them, and yes, so, the forcing was unnatural, totally unnatural. It is worth noting that one person and one person only had been entirely convinced that CH was going to be proven independent, and that person was Gödel. In spite of his own proof of the relative consistency of CH he wrote as early as 1947 that CH is most likely to be independent.*[21]

D. Kant: *And then people quickly understood this technique and applied it?*

M. Džamonja: *Some people did. Yes, it started in California and well, there was Solovay, and there were also Dana Scott and there were many other people there around Stanford and Berkeley, for example Ken Kunen and Bill Mitchell among the younger oner. Paul Cohen was in Stanford and Ken was a student in Stanford. So, I think, the forcing was localized to the United States for a while. But then just a year or two later, it spread around to Israel. Yes, people did understand, but I think, it was rather slow. I mean, the specialists understood perhaps quickly but it was slow and it was not published, Cohen's book*[22] *took time to be published and it is not easy to learn the method from this book.*

D. Kant: *What would you say, when or with which results did forcing become more natural?*

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[20] Amazingly, this was not the case of Gödel, who shortly after discovering L and proving the relative consistency of GCH stated that he believes in the independence of GCH, see the beginning of the interview for this. But then, Gödel was considered a logician and a philosopher, not a mathematician, by the peers of the time.

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M. Džamonja: *I think a subject generally becomes natural when people start writing and reading books about it. In this case it was quite late, for example Kunen's Set Theory*[23] *came out, in 1980, and that is really where people learned this from, from a book. Jech's book*[24] *also came out at that time. Before that, well, if you were at the right place at the right time you learned something about it. But it wasn't a well spread method. For example, I came from a country [Yugoslavia] in which there was a considerable amount of set theory, combinatorial set theory. But nobody was really doing forcing. I finished my undergraduate degree in 1984 and I wanted to write a thesis (we needed to write a thesis at the end of our undergraduate studies) on forcing. But I couldn't find an advisor for this in Sarajevo. So, I think this tells you that people knowing this subject were rare, from the Yugoslav perspective. Maybe Kurepa knew it, in Belgrade. I don't even know if even he learned this method, I don't think he published any papers of this. And he was probably one of the, greatest set theorists of the previous generation. In Hungary, I think, it took quite a while, it was maybe only when Soukup and Peter Komjath worked on this that it was seriously understood, and it was in the 80s. So, it took some time. In other countries of Eastern Europe there were people like Bukovsky and Balcar who worked on this already in the 1960s, but it was politically difficult for them. Their work was practically unknown to others because of the Cold War. And finally, in Russia, this method just didn't came through. Moti Gitik came from Russia to Israel, thinking that he had discovered a new method, the method of forcing. He discovered it on his own, because he didn't have access to the research already known in the West. Probably for someone of your generation, it's very difficult to imagine that time.*

D. Kant: *Yes.*

M. Džamonja: *But literature was a really big problem. The Cold War for one, and the other thing was the cost of journals. It was incredibly expensive. Except for top universities you wouldn't find in your university library journals that publish this kind of thing. So, it was really very restricted.*

We see that there was not as much resistance to adopt the forcing method, as there was a kind of disappointment and resignation on the one hand, interest and enthusiasm on the other hand, and circumstances of that time which made the adoption of forcing a slow process. Nowadays, every set theorist knows of the forcing method, though not everyone works with it. The forcing technique has been developed for application to many problems.

Important Subsequent Forcing Results

This subsection contains more mathematical details than the other parts. We present important conjectures, theorems, and forcing axioms, such as Easton's theorem,

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Suslin's Hypothesis, Martin's Axiom, the Borel Conjecture, and the Proper Forcing Axiom, and we define many notions involved . It is intended to illustrate the set-theoretic research on independence.

With Cohen's original method, one can prove that $2^{\aleph_0}$, which is the size of the continuum, can be any regular uncountable cardinal (and even more generally, any cardinal of uncountable cofinality). A regular cardinal κ is one which cannot be obtained as a supremum of a sequence of cardinals of length less than κ many. On the contrary, asingular cardinal κ is one that can be reached in less than κ many steps. For example, the cardinal $\aleph_\omega$, which is greater than ω, can be reached in ω many steps: $\aleph_0, \aleph_1, \aleph_2, ..., \aleph_\omega$.[25] So, given an uncountable regular cardinal κ, one can force the statement '$2^{\aleph_0} = \kappa$'.

After Cohen presented this method, he did however not pursue to work on different applications of it. Instead, other people were learning and applying the forcing method, and thus solved many open problems. Robert Solovay was one of the most important pioneers in forcing from the early 60s on. One should of course also mention Jack Silver and several other important set theorists from that time.

Easton Forcing

Solovay also had many excellent students, including Matthew Foreman and Woodin. An early student was William Easton, who introduced in his PhD thesis a new kind of forcing. To show that the GCH can be violated almost arbitrarily, he used a proper class[26] of forcing conditions, where Cohen forcing only uses a set of forcing conditions. The only restrictions for cardinalities of 2^κ for regular κ are given by the requirement that $\kappa < \lambda$ implies that $2^\kappa \leq 2^\lambda$, and by König's Theorem:

König's Theorem (for cofinalities), 1905: *For every cardinal κ, it holds $cf(2^\kappa) > \kappa$.*[27]

Then Easton was able to present what is now called *Easton Forcing* and proved the following theorem.

ZFC + ¬GCH, Easton's Theorem, 1970: *Let F be a non-decreasing function on the regular cardinals, such that $cf(F(\kappa)) > \kappa$ for every regular κ. Then, by a cofinality*

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[25] The correct notion to distinguish between regular and singular cardinals is the notion of cofinality. The cofinality of a cardinal κ is the length of the shortest sequence of ordinals less than κ which converges to κ.

[26] A proper class contains all sets that satisfy a given first order formula, but is itself not a set. (Thus, a proper class is defined by unrestricted comprehension over the universe of sets.)

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and cardinality preserving forcing, we can obtain a model in which $F(\kappa) = 2^\kappa$ for every regular κ.[28]

Suslin's Hypothesis, iterated Forcing and Martin's Axiom

An old hypothesis about properties of the real line became very important in set theory. It is based on a question asked by Michail J. Suslin.[29]

Suslin's Hypothesis (SH), 1920: Every dense, linear order, in which there are at most countably many disjoint open intervals, is isomorphic to the real line.

One can also define a Suslin Tree, and then the Suslin Hypothesis states that there is no Suslin Tree. This version is often used in practice.[30]

Suslin's Hypothesis is independent of ZFC. It was shown, independently by Stanley Tennenbaum in 1968 and Thomas Jech in 1967 (see Jechs book **?** for historical remarks and the references), that SH cannot be proved in ZFC. Also in L, it is false, which was proved by Ronald Jensen. He proved that the axiom V=L implies the $\diamondsuit$-principle:[31]

$\diamondsuit$**-principle**, 1972: There exists a sequence of sets $\langle S_\alpha : \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$, such that for every $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1.[32] The sequence $\langle S_\alpha : \alpha < \omega_1 \rangle$ is called a $\diamondsuit$*-sequence*.[33]

Jensen proved that $\diamondsuit$ implies that there is a Suslin Tree:

ZFC $+ \neg$**SH**: In ZFC+V=L, one can prove the existence of a Suslin Tree.[34]

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28 (**?**, Theorem 1, pp.140-1). It is interesting to note that this is potentially class forcing. The Forcing Theorem of Cohen only applies to special cases of class forcing, so class forcing is is less widely spread in applications. All of the forcing notions to follow will be set forcings.

29 "3) Un ensemble ordonné (linéairement) sans sauts ni lacunes et tel tout ensemble de ses intervalles (contenant plus qu'un élément) n'empiétant pas les uns sur les autres est au plus dénombrable, est-il nécessairement un continu linéaire (ordinaire)?" **?**. Translation (by the authors): "Is a (linearly) ordered set without jumps nor gaps, such that every set of its invervals (containing more than one element) which do not overlap each other is at most countable, necessarily a linear continuum?".

30 *See* (**?**, pp.114-116).

31 'Diamond-principle'

32 A subset of ω_1 is called *stationary* if it intersects all closed and unbounded subsets $C \subseteq \omega_1$, where C is *closed* if for every sequence $(a_n)_{n<\omega} \subseteq C$ the limit $\bigcup\{a_n : n < \omega\}$ is also an element of C, and C is *unbounded* if for every $a \in C$ there is a $b \in C$ such that $b > a$.

33 (**?**, p.191)

34 Jensen showed actually a more general version, from which this theorem is one instance (**?**, Theorem 6.2 and Lemma 6.5, pp.292-5).

For the complementary result, Solovay and Tennenbaum showed by a newly introduced kind of forcing that there is a forcing extension of ZFC+SH. They used *iterated Forcing*, which simply means that single forcing notions are applied, applied again, and again, and so on, transfinitely many times. To make an iteration of forcing notions work, one needs a *preservation theorem* which guarantees that the iteration of single well-behaved forcings is still well-behaved:

> Roughly spaking, it [the preservation theorem] says that the transfinite iteration of a sequence of Cohen extensions satisfies the countable chain condition (c.c.c.) if every stage satisfies c.c.c.[35]

Well-behaved means here *satisfying the countable chain condition*.

To define this condition, we need to know that a *forcing notion* $(P,<)$ is a partial order, that two elements $p,q \in P$ are called *incompatible* if there is no $r \in P$ such that $p < r$ and $q < r$, and that an *antichain* is a set $A \subseteq P$ such that all of its elements are pairwise incompatible.

Countable chain condition (c.c.c.): A forcing notion $(P,<)$ satisfies the countable chain condition, if every antichain is at most countable.

To call a c.c.c. forcing notion well-behaved is explained by the fact that c.c.c. forcing notions preserve cardinals and cofinalities. In general, when using an arbitrary forcing notion, many things can happen that are undesired. Cardinals can be collapsed, cofinalities can be changed etc. and then, the desired statement might not be forced. Therefore, set theorists often consider certain well-behaved forcing notions, which, for instance, preserve cardinals and cofinalities.

Now, iterating certain c.c.c. forcing notions permitted Solovay and Tennenbaum to construct a forcing extension in which ZFC+SH holds, and also a forcing axiom called Martin's Axiom (named after Donald A. Martin):

Martin's Axiom (MA): For every c.c.c. forcing notion $(P,<)$ and every family of dense sets $\mathscr{D}$ such that $|\mathscr{D}| < 2^{\aleph_0}$ there exists a $\mathscr{D}$-generic subset $G \subseteq P$.[36]

Given CH, i.e. $2^{\aleph_0} = \aleph_1$, Martin's Axiom is provable, because in this case $\mathscr{D}$ can only be countable, and for countably many dense sets, there is always a generic set. Therefore, when using Martin's Axiom, it is often additionally assumed that $2^{\aleph_0} > \aleph_1$. In a footnote of their article, Solovay and Tennenbaum explain that first, they worked without Martin's Axiom but that Martin noticed a possible formulation in axiomatic terms.[37] However, they prefer using Martin's Axiom because the axiom gives rise to a general proof scheme that is easier to apply than the repeated iteration of c.c.c. forcings.

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[35] **?**, they refer to theorem 6.3 on p.228.

[36] G is $\mathscr{D}$-generic means that $G \cap D \neq \emptyset$ for every $D \in \mathscr{D}$.

[37] (**?**, fn on p.232)

> Most of the applications of iteration to date may be presented in the following manner. One shows that M [MA] (or possibly M + "$2^{\aleph_0} > \aleph_1$") implies a theorem T. Then the consistency proof for ZF + AC + M yields a consistency proof for ZF + AC + T.

What is needed, to follow this approach, is a relative consistency proof for Martin's Axiom, which is also given by Solovay and Tennenbaum. But once shown that MA is consistent, later applications only need to show that MA implies some theorem T, and then, the result is that T is relatively consistent to ZFC. They apply this scheme to SH and prove

ZFC + SH: "*Suppose ZF is consistent. Then so is ZF + AC + SH.*"[38]

Laver Forcing

Laver Forcing was developed to show the independence of the Borel Conjecture.

Borel Conjecture, 1919: Every strong measure zero set is countable.[39]

Wacław Sierpiński proved in 1928 that the Borel Conjecture can be false.[40] To show that the Borel Conjecture can also be true, Richard Laver developed a forcing to add specific reals (*Laver reals*), and iterated this forcing.

ZFC + Borel Conjecture, Laver Forcing, 1976: "*If ZFC is consistent, then so is ZFC+Borel's Conjecture.*"[41]

Proper Forcing and Proper Forcing Axiom

Proper Forcing was defined by Saharon Shelah.[42] He takes proper forcings to be well-behaved forcings, in particular because they behave nicely when they are iterated:

> When we iterate we are faced with the problem of obtaining for the iteration the good properties of the single steps of iterations. Usually, in our context, the worst possible vice of a forcing notion is that is collapses $\aleph_1$. The virtue of not collapsing $\aleph_1$ is not inherited by the iteration from its single components. As we saw, the virtue of the c.c.c. is inherited by the ... iteration from its components. However in many cases the c.c.c. is too strong a

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[38] (**?**, Theorem 7.11 on p.242)

[39] **?**. A *strong measure zero set* is a subset X of the reals such that for every sequence $\langle \varepsilon_n : n < \omega \rangle$ of positive real numbers there is a sequence $\langle I_n : n < \omega \rangle$ of intervals with length$(I_n) \leq \varepsilon_n$ such that $X \subseteq \bigcup\{I_n : n < \omega\}$.

[40] **?**

[41] (**?**, Theorem on p.152), *see* also (**?**, pp.564-8).

[42] **?**

> requirement. We shall look for a weaker requirement which is more naturally connected to the property of not collapsing $\aleph_1$, and which is inherited by suitable iterations.[43]

The weaker requirement Shelah is looking for is properness.

Proper Forcing: A forcing notion $(P,<)$ is *proper* if forcing with P preserves for every uncountable cardinal λ the stationary sets of $[\lambda]^\omega$.[44]

Many specific forcing notions can be shown to be proper. For instance, every c.c.c. forcing is proper.[45]

As for c.c.c. forcing, one also needs a preservation theorem to iterate proper forcing, which is an important result by Shelah.[46] Since iterating proper forcing works well, an axiom was formulated to simplify using this forcing.

Proper Forcing Axiom (PFA): For every proper forcing notion $(P,<)$ and every family of dense sets $\mathscr{D}$ such that $|\mathscr{D}| = \aleph_1$ there exists a $\mathscr{D}$-generic subset $G \subseteq P$.

Again, once the relative consistency of the forcing axiom is shown,[47] one can prove theorems as consequences of the forcing axiom and circumvent to work out the details of the iterated forcing method at every application. The consequences of a forcing axiom are then proven relatively consistent to ZFC. One important consequence of the proper forcing axiom was proven by Boban Veličković and Stevo Todorčević:

In **ZFC + PFA**: *Assuming the Proper Forcing Axiom, one can prove* $2^{\aleph_0} = \aleph_2$.[48]

This means, that in ZFC+PFA the Continuum Hypothesis is false.

Today, most forcings applied in practice are iterated forcings or the forcing axioms obtained by consistency proofs through iterated forcing. For example, a current research question is to find properties of forcing notions which allow the existence of set-theoretic universes saturated for the generics for dense sets of size $\aleph_2$.

4 Philosophical Thoughts in Set Theory

For someone who is interested in philosophical questions, set theory is an exciting subject matter. Questions that arise there are for example the truth question: Are the

43 (**?**, p.90)
44 $[\lambda]^\omega$ is the set of all countable subsets of λ.
45 (**?**, Lemma 31.2 on p. 601)
46 *See* (**?**, III. §3.)
47 Assuming that there is a supercompact cardinal, one can prove that there is a model of ZFC+PFA.
48 (**?**, Theorem 31.23 on p. 609)

independent sentences in set theory neither true nor false? What is truth in set theory if it does not coincide with provability? The truth question is part of the traditional philosophy of mathematics, as well as the question whether sets exist. If we assume that sets exist then we can easily give an account of truth: A sentence is true if and only if it is true in the universe of sets.

A close, but not the same, definition is the following: A set-theoretic sentence is true if and only if it is true in the intended model of set theory. This definition does not presuppose the existence of a universe of sets because it refers to the technical term of an intended model. Such a model may be given formalistically, i.e. as part of a formal theory. This definition is for example usually used when considering the formal number theory PA.[49] The standard model $\mathfrak{N} = (\mathbb{N}, 0, S, +, \cdot, <)$ of number theory can be given in set-theoretic terms. So, it can be given in the formal theory ZFC. From a formalistic point of view, one could adopt the second definition (if one believes that there is an intended model) but not the first. And from the point of view of a platonist, the first and the second definition would correspond to each other—the reality (not just the intended model) would be taken as the formal counterpart of the universe of sets.

Such ideas about truth and existence are some of the philosophical thoughts that we find in set-theoretic practice. They are revealed for example as a justification of axioms. When the second author asked the first author about the meaning of the word "axiom", she had a clear answer: "For me, what it means is an obvious property of the intended universe," and she admitted that this "is a strong meaning because it implies the existence of an intended universe." This corresponds exactly to the above mentioned view of true set-theoretic sentences. But the word is also used for the new axioms, which are either not generally accepted by set theorists or their acceptance is less obvious than the acceptance of the ZFC-axioms. How is the word "axiom" used in this context?

M. Džamonja: *When we think of axioms in the classical sense, we think of Euclid, and his geometry, and the idea there is that the axioms are statements that are obvious. Obvious in the sense that we take some basic objects, which I think in Euclid's mind come from his intended application which he takes as the only one, and then the axioms are certain statements that are obviously true about these basic objects. From these we build out further content. I think the idea of axiomatic set theory was to do this but with mathematics in general. The basic objects are sets. So, certainly the Axiom of Pairing is obvious, even though now with the Homotopy Type Theory it's a complicated issue, but in the classical set theory, this type of axioms—Pairing, Union—are somehow clear. Well, some of the classical axioms are also less clear, of course an example is the notorious Axiom of Choice. It is not clear in what sense it is an axiom. And in fact maybe, that situation between ZF and ZF with Choice is in some sense similar to the situation between ZFC plus some Forcing Axiom or just ZFC because you might say that for some people the Axiom of Choice was not natural, they refused to*

[49] PA stands for Peano Arithmetic which is the formal theory of the natural numbers.

work with it and they worked in ZF set theory. Or even, you can say that for those people working with the ZFC set theory it is also interesting to understand where one really needs the Axiom of Choice, so to work somehow between ZF and ZFC. Certainly then one doesn't have to take an opinion of whether the Axiom of Choice is true philosophically or not, but can work in both ZF and ZFC and somehow take the neutral view that this is what I can do if I have the Axiom of Choice and this is what I can do otherwise.
So, if you take that view, then the Forcing Axioms are consistent extensions of ZFC. The first Forcing Axiom —the Martin Axiom– has the same consistency strength as ZFC, we do not need any extra Large Cardinal Axiom to prove its consistency. So, if we just concentrate on that one, so if we look at ZFC vs. ZFC plus Martin Axiom, then we can say, well, we don't have to take the view if Martin Axiom is true or not, but we could say that this is a possible axiom to add. Is this true or not, well, we don't know. So, in that sense, it is reasonable to makes this an axiom. Also, it was the first forcing-related statement that got in any way close to being, let us say, comprehensible to a large number of mathematicians and logicians. Once we had the Martin Axiom, of course, the extensions of it started coming, like Proper Forcing Axiom. They are extensions because, mathematically speaking, they are very similar to the Martin Axiom. Logically speaking, they are not extensions, because they require Large Cardinal Axioms, so they lose that property of equiconsistency with ZF that we had before. Considering ZFC or ZFC plus Proper Forcing Axiom for example, we have a much stronger consistency strength with PFA added. So the two are not exactly at the same level. In the end, by extending the strength of these Forcing Axioms, we seem to get further and further from what an actual axiom might mean.

We have seen in these paragraphs, that the word "axiom" can also be used as a rather technical term without any philosophical implications. This is also possible regarding the ZFC-axioms, but it is even more important to emphasise the possibility of such a *neutral view* regarding the new axioms. In this view, the use of the word "axiom" does not imply that this statement is in any way accepted as a statement itself. It is rather accepted as a reasonable, possible statement that could be added to ZFC. Thus, one can work in the corresponding theory to address mathematical questions without taking a stance on its acceptability.

But still, the answer that an axiom is either obviously true or just a possible statement to assume, does not seem to give the whole story. The role of Large Cardinal Axioms in set-theoretic practice could be a challenge to this view. These axioms are not accepted similar to the ZFC-axioms, but they are also not merely possible axioms to add for certain purposes. For instance, every Large Cardinal Axiom (at least as strong as the existence of an inaccessible) implies the consistency of ZFC, and this seems partially to support their acceptability.

D. Kant: *What would you say about the statement "ZFC is consistent"? Does it still play a role in set-theoretic practice?*

M. Džamonja: *Well, of course, in the beginning it was hoped that we will prove that ZFC is consistent. That was Hilbert's Program,*[50] *and then Gödel's results said, if we believe that ZFC is the basic theory, then we cannot, within that basic theory, prove that it is consistent. So now, we have two choices: Either we accept just ZFC as our basic theory and then we have to take on faith that it is consistent, or we say, well ok, I believe in large cardinals, and then I get the consistency of ZFC for free; in the sense that, when we take the cumulative hierarchy and cut it at a large cardinal, we get a model of ZFC.*
This is actually the other side of your question of what is an axiom. If we have an axiom scheme that is supposed to be obvious, then it is supposed to be talking about the intended model. Now, there are people who do not believe in the intended model. I believe that there is a universe of sets, personally, this is my philosophical view. So, if there is such a thing, then the ZFC Axioms are the axioms of this universe. They are not the only axioms but they are the axioms that we accept. They describe this universe quite well, so, they have an intended model, they have other models as well. But, somehow believing in the consistency comes back to thinking if there is this universe of sets or not. And, I think, this is now a philosophical question rather than a mathematical one.

D. Kant: *So, you would also think that among set theorists, the existence of V,—I mean, this candidate for the intended model is somehow subjective, and some believe in it, some do not...*

M. Džamonja: *Yes, for example, I think, Gödel was a very strong Platonist. Woodin confirms to be a very strong Platonist and he is searching for more complete axioms of set theory. I think, Shelah also is a Platonist. But I know people, like Cummings for example, who told me some years ago, that for him, the question if there is an intended model or not is not at all interesting. What is interesting is that we get to do beautiful mathematics with these objects and if they exist or not, is not that interesting. So, one can do the same mathematics independently of one's philosophical view. In fact, mathematicians in general, even set theorists who work in logic, do not always ask philosophical questions. Some do and some don't.*

In set theory, there are some mathematicians who think about philosophical concepts such as platonism. However, it is wrong to say that every set theorist thinks about independence also in philosophical terms. Džamonja suggests that it is a matter of interest. There may be more set theorists who are interested in philosophical questions than there are more general mathematicians, but this is not the case for all set theorists.

In the above conversation, the possibility to believe in large cardinals is mentioned. This highlights an important difference between the Large Cardinal Axioms and the Forcing Axioms. We think that Forcing Axioms are mostly not seen as can-

[50] Hilbert wanted to prove the consistency of mathematics, and focussed on axiomatisations of number theory. His program can be transferred to set theory, as set theory counts as a foundation of mathematics, and if one had proved its consistency, Hilbert's aim would have been resolved.

didates for acceptance.[51] The function of a Forcing Axiom is not to capture a possible truth about the universe of sets, but rather to formalise a specific and fruitful kind of forcing. The formulation of such an axiom makes the application of forcing easier, because one does not have to build up the whole forcing machinery each time.

M. Džamonja also says explicitely that she believes in the existence of a universe of sets. She said that she believes the ZFC Axioms as well as the Large Cardinal Axioms. However, she made clear that this belief notion is indeed a relative one. Since the universe of sets is an abstract reality, it is possible that it is not the unique reality for all of mathematics. This view is recently supported by the research on the Univalent Foundations and Homotopy Type Theory (HoTT). This mathematical field has its own concepts and methods which differ significantly from other mathematical fields; it creates own content, and mathematics can be embedded in this theory. However, some set-theoretic principles do not generally hold there, e.g. the Axiom of Choice. With this in mind, M. Džamonja believes in the Axiom of Choice, but only restricted to set theory, not in general for all of mathematics. This implies that set-theoretic reality can well embed much mathematics, but HoTT can as well, and each field has its own reality in this platonistic view.

5 Set-theoretic intuition about independence

The truth question as well as the existence question mentioned in the last section are part of the classical philosophy of mathematics. There is much literature on these topics, however these questions are not those which the second author searches to answer by talking to set theorists. One can observe that in set theory, one independence result is not similarly conceived of as another; set theorists see differences in the value of the insights which they provide, or in the naturalness of independent statements. For example, some set theorists consider the axiom $V = L$ to be less natural than the existence of infinitely many Woodin cardinals, because $V = L$ is not compatible with the existence of Woodin cardinals. Woodin says: "[T]he axiom $V = L$ limits the large cardinal axioms which can hold and so the axiom is *false*."[52] The existence of infinitely many Woodin cardinals imply Projective Determinacy, which is an attractive statement for some set theorists, for example for Ralf Schindler: "The principle of projective determinacy, being independent from the standard axiom system of set theory, produces a fairly complete picture of the theory of "definable" sets of reals."[53]

In order to elaborate on such judgements, we were talking about the question whether set theorists have an intuition about their subject matter which is based on

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[51] Menachem Magidor certainly is an exception because he thinks that Forcing Axioms are natural axioms.

[52] His italics, (**?**, p.504)

[53] www.math.uci.edu/node/20943

their wide experience, but which is not necessarily backed up by proofs.

D. Kant: *I imagine that set theorists have gained a good intuition about what is provable in ZFC and what is independent. Would you say that you have a good intuition about this?*

M. Džamonja: *Yes, I think we do have a good intuition. Of course not about everything but about certain things, certain areas. I have a way that I see it, I think of the line of cardinal numbers. Then certain areas of that line are well understood, and we really have an intuition in that context, but other areas are murky.*

D. Kant: *Is there maybe something that you can say about these borders of ZFC? More concretely, is there something about these independent sentences that they have in common?*

M. Džamonja: *To start with, there are certain things that definitely cannot be independent because they are described by simple formulas and we have absoluteness theorems.*[54] *Things that are combinatorially close to them can likely be shown to be true or not true. So descriptive set theory and things that go with it. We may find there some sort of mini-independence. They would be connected with certain classes of sets whose properties would be understood within ZF bit which exhibit less aboslutness, such as analytic sets, or projective sets etc.*

Sometimes we can reflect independence to truth by restricting our attention to certain classes of sets. For example, suppose that we can use a Forcing Axiom to prove some statement about subsets of the reals in general. We can then hope to have the same statement hold about analytic sets without needing any additional axioms. Certain results that are obtained under PFA for general sets turn out to be true for analytic sets. For example, one can find this in the work of Todorcevic about gaps.[55] *There are analytic gaps, there are general gaps, there is the p-ideal dichotomy, and then there is this dichotomy applied to analytic objects, or in the work of Solecki. So, that is one border of independence.*

Another border is, as I mentioned, the line of cardinal numbers. We know that at successors of regular cardinals we can do a lot by forcing, especially at $\aleph_1$. We also know that at singular cardinals and their successors, things are much more, let us say, resistant to forcing. This is so because we have pcf theory which shows that some things about singular cardinals, are just true in ZFC and therefore, many statements that are implied by pcf theory are also just true.[56] *So, there are these two distinct regions on the cardinal numbers line. There are successors of regular cardinals which have some behaviour and then there are singular cardinals and their successors which have another one, and then of course, there are large cardinals*[57]*. So, we do have a good intuition when we start from a certain*

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[54] For every Δ_0-sentence φ (a sentence with only bounded quantifiers), and every transitive standard model M of ZFC, it holds ZFC $\vdash \varphi \Leftrightarrow M \models \varphi$. This means that Δ_0-sentences cannot be independent. They are either provable or refutable. There are other well-known absolutness theorems, such as the Schoenfield's Absolutness.

[55] See for example **?** and **?**

[56] **?**, **?**

[57] For a philosophical discussion of these regions see **?**

kind of cardinal in which direction to try to start working. And then we also have a good intuition about the kind of sentences as explained above. Combinatorial set theory is almost always about unrestricted sets, so there, we can expect to have a lot of independence.

D. Kant: *About independent sentences in history, have there been some surprising, or unexpected results? So, that at first, the sentence was thought to be true, and then it turned out to be independent, or something like that? That really set theorists ...*

M. Džamonja: ... *were surprised?*

D. Kant: *Yes, were surprised about what came out.*

M. Džamonja: *Well, we have already said that the independence of CH came as a surprise to many mathematicians. But there is a recent example of just the opposite, when a statement was thought to be independent but at the end it turned just to be true. I refer to a theorem by Malliaris and Shelah.*[58] *They proved a certain cardinal equality, that is, they proved that two cardinals,* $\mathfrak{p}$ *and* $\mathfrak{t}$, *which are cardinal invariants of the continuum, are actually just equal in ZFC. This was totally unexpected because there are many cardinal invariants known and they are tend to be independent from each other. The independence of* $\mathfrak{p}$ *and* $\mathfrak{t}$ *was one of the last open questions and everybody expected they would behave like any other invariants, be independent—but they are not! The Maliaris-Shelah proof is also very ingenious, it mixes many different methods. That proof obtained an important prize in 2017, the Hausdorff medal that is given biannually by the European Set Theory Society for the most influential work published in the last five years.*

D. Kant: *But this does not happen very often?*

M. Džamonja: *No. That does not happen very often. Well see, what I think is that in mathematics, a huge percentage of results is proving something that is not so surprising. Everywhere in mathematics, including set theory, there are results that everybody suspects to hold, but if you want to be sure, you have to produce a proof. So, when somebody takes two new cardinal invariants and makes them independent of each other, that makes an ok PhD thesis but it does not make a huge surprise or a Hausdorff medal. Because we have seen such results very often. The opposite is surprising.*

This conversation should make clear that set theorists can say something more about the independence phenomenon than what they can prove. They can sometimes give probabilistic statements about the mathematical objects they work with. One example was mentioned: Two cardinal invariants are often independent. Of course, such general ideas are based on experience and could turn out to be wrong—just imagine that there will be found many cardinal invariants such that pairs of them can often shown to be equal. However, such probabilistic, experience-based statements seem to play a very important role in set-theoretic research. In addition, it would be

[58] **?**

an interesting philosophical question whether or how such general ideas can be seen as a part of set-theoretic knowledge.

For a first exploration on this question, we would argue that experience-based statements are part of set-theoretic knowledge. Given the possible change in the future, they would have to be relativised to a time frame in which they correspond to beliefs of most set theorists. Describing this part of set-theoretic knowledge is most valuable because it can explain the development of the discipline. Imagine that some day, a new axiom is accepted, then this would be possibly seen as a surprising when only looking at the theorems. However, it could possibly be easily explained when looking at the experience-based, probabilistic part of set-theoretic knowledge. Furthermore, also normative judgements by set theorists play an important role in this context, which can as well be hoped to be easily explained by the informal part of set-theoretic knowledge as we aimed at grasping here.

6 Conclusion

We started with a philosophical perspective on the set-theoretic independence phenomenon. In philosophical terms, this mathematical phenomenon raises many questions and can appropriately be called independence *problem*. In mathematical terms, it is not clear whether the independence phenomenon is a problem or a mere mathematical fact.

Putting together our mathematical and philosophical perspectives, we gave an overview of contemporary set theory, in which we focussed on forcing in order to illustrate to what large extent independence results determine today's research. After that, we presented Džamonja's view on various topics such as the introduction of forcing in set theory, the use of new axioms in practice (distinguishing between ZFC-axioms, forcing axioms, and large cardinal axioms), possible decidability of independent sentences, and surprising events. What we wanted to grasp with that is what Tao calls the "solid intuition"[59] of an expert mathematician on her/his field of expertise. This solid intuition is founded in many years of set-theoretic research, which seems to make it unavailable to non-set theorists. For, it is often the set-theoretic formalism and rigour which make it hard for philosophers and other mathematicians to obtain an idea of the topic in set-theoretic research. However,

> it is only with a combination of both rigorous formalism and good intuition that one can tackle complex mathematical problems; one needs the former to correctly deal with the fine details, and the latter to correctly deal with the big picture.[60]

In these terms, we can hardly hope to widely communicate the fine details since this would require set-theoretic training, but we can hope to communicate a big picture in a comprehensible and correct way. Such a big picture is expected to include dif-

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59 **?**

60 ibid.

ferent perspectives of set theorists which differ on some aspects, and which agree on others. It would be correct if it is consistent with set-theoretic practice.

To close, we want to leave the reader with a question and three hypotheses.

Question: What are the different aims/motivations for the uses of different axioms?

1. Hypothesis: Most set theorists were surprised by the introduction of the forcing method.
2. Hypothesis: Most set theorists think that forcing is a natural part of contemporary set theory.
3. Hypothesis: Most set theorists prefer an answer to a problem with the help of a new axiom of lowest possible consistency strength. And for most set theorists, a difference in consistency strength weighs much more than the difference between forcing axiom and large cardinal axiom.

With regard to different uses of new axioms, we distinguished a neutral view for the use of forcing axioms on the one hand, and the use of accepted large cardinal axioms on the other hand. This analysis can certainly be refined. The hypotheses are motivated because they correspond to Džamonja's view which we think is representable for other set theorists as well. However, there could be objections raised. In contrast to the first thesis, one could also support a historical view according to which the time was ripe for the forcing method. The second thesis may be challenged by views of descriptive set theorists who rarely use forcing in their research. And for the third thesis, we may find set theorists who would not be willing to agree (for example, for many years the school of the set theory of the reals did not accept the idea of large cardinals). Thus, we are not in the position to draw final conclusions. Rather, we encourage further research on the practice which will bring further clarification.

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