Homotopical and homological finiteness properties of monoids and their subgroups

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Fact: There are lots of nasty finitely presented monoids out there.

Markov (1947), Post (1947): There exist finitely presented monoids for which there is no algorithm to solve the word problem.

Idea

1. Identify a class $C$ of “nice” finite presentations:
   - finite complete rewriting systems
     - noetherian and confluent
   
   A monoid defined by a finite complete rewriting system has solvable word problem.

2. Try to gain understanding of those monoids that may be defined by presentations from $C$:
   - study properties of monoids defined by such rewriting systems:
     - Finite derivation type (FDT)
     - $FP_n$
Finite derivation type
a homotopical finiteness condition

- Is a property of finitely presented monoids.
- Introduced by Squier (1994)
  - (and independently by Pride (1995))

Original motivation
To capture much of the information of a finite complete rewriting system for a monoid in a property which is independent of the choice of presentation.

- Connections with diagram groups (which are fundamental groups of Squier complexes of monoid presentations)
  - Kilibarda (1997)
  - Guba & Sapir (1997)
The derivation graph of a presentation

- $\mathcal{P} = \langle A \mid R \rangle$ a monoid presentation
  - $A$ - alphabet, $R \subseteq A^* \times A^*$ - rewrite rules
- **Derivation graph:** $\Gamma = \Gamma(\mathcal{P}) = (V, E, \iota, \tau, \iota^{-1})$:
  - **Vertices:** $V = A^*$
  - **Edges are 4-tuples:**
    $\{(u, r, \epsilon, v) : u, v \in A^*, r = (r_{+1}, r_{-1}) \in R, \text{ and } \epsilon \in \{+1, -1\}\}$. 
- **Initial and terminal vertices:** $\iota, \tau : E \rightarrow V$ for $E = (u, r, \epsilon, v)$:
  - $\iota E = ur_\epsilon v$, $\tau E = ur_{-\epsilon} v$
- **Inverse edge mapping:** $\iota^{-1} : E \rightarrow E$
  - $(u, r, \epsilon, v)^{-1} = (u, r, -\epsilon, v)$.
Paths and pictures

Example. \( \langle x, y | xy = y, yx^2 = y^3 \rangle \)

A path is a sequence 
\( \mathbb{P} = \mathbb{E}_1 \circ \mathbb{E}_2 \circ \ldots \circ \mathbb{E}_n \) where 
\( \tau \mathbb{E}_i \equiv \iota \mathbb{E}_{i+1} \).

Gluing edge-pictures together we obtain pictures for paths.

\( \iota \) and \( \tau \) can be defined for paths

In this example 
\( \iota \mathbb{P} = yxyxxxx, \ \tau \mathbb{P} = yyxxyy. \)
Operations on pictures

\[ \mathcal{P} = \langle A | R \rangle, \quad \Gamma = \Gamma(\mathcal{P}) \]

Pictures \iff Paths

- **Parallel paths:** write \( \mathcal{P} \parallel \mathcal{Q} \) if \( \iota \mathcal{P} \equiv \iota \mathcal{Q} \) and \( \tau \mathcal{P} \equiv \tau \mathcal{Q} \).
- **X** - set of pairs of paths \((\mathcal{P}_1, \mathcal{P}_2)\) such that \( \mathcal{P}_1 \parallel \mathcal{P}_2 \).

**Idea**

Want to regard certain paths as being equivalent to one another modulo \( X \).
Operations on pictures

Basic operation (I): Deleting a cancelling pair

Basic operation (II): Interchanging disjoint discs
Operations on pictures

Basic operation (III): Replacing a subpicture using $X$
Replace a subpicture $P_1$ by $P_2$ provided $(P_1, P_2) \in X$. 

\[ u \cdot P_1 \cdot v \xrightarrow{\sim} u \cdot P_2 \cdot v \]
Homotopy bases

Note: Applications of these picture operations do not change the initial vertex or the terminal vertex of the original path.

A homotopy base is...

a set $X$ of parallel paths such that given an arbitrary pair $(P_1, P_2) \in \parallel$ we can transform $P_1$ into $P_2$ by a finite sequence of elementary picture operations (and their inverses)

(I) cancelling pairs,  (II) disjoint discs,  (III) applying $X$. 
Finite derivation type

**Definition**

$\mathcal{P} = \langle A | R \rangle$ has **finite derivation type (FDT)** if there is a **finite homotopy base** for $\Gamma = \Gamma(\mathcal{P})$. A monoid $M$ has FDT if it may be defined by a presentation with FDT.

**Theorem (Squier (1994))**

- *The property FDT is independent of choice of finite presentation.*
- *Let $M$ be a finitely presented monoid. If $M$ has a presentation by a finite complete rewriting system then $M$ has FDT.*
Monoids and their subgroups

Idea

Relate the problem of understanding a property for monoids with the problem of understanding the property for groups.

- $M$ - monoid
- Green’s relations $\mathcal{R}$, $\mathcal{L}$, and $\mathcal{H}$
  
  \[ x \mathcal{R} y \iff xM = yM, \quad x \mathcal{L} y \iff Mx = My, \quad \mathcal{H} = \mathcal{R} \cap \mathcal{L}. \]

- $H$ = an $\mathcal{H}$-class. If $H$ contains an idempotent $e$ then $H$ is a group with identity $e$.
  
  ▶ These are precisely the maximal subgroups of $M$.

General question: How do the properties of $M$ relate to those of the maximal subgroups of $M$?
Finite derivation type for subgroups of monoids  
(joint work with A. Malheiro)

Theorem

Let $M$ be a monoid and let $H$ be a maximal subgroup of $M$. If the $\mathcal{R}$-class of $H$ contains only finitely many $\mathcal{H}$-classes then:

- $M$ has FDT $\Rightarrow$ $H$ has FDT.
- Given a homotopy base $X$ for $M$ we show how to construct a homotopy base $Y$ for $H$. Finiteness is preserved when the $\mathcal{R}$-class has only finitely many $\mathcal{H}$-classes.
- **Ruskuc (1999):** Proved the corresponding result for finite presentability.
A semigroup is regular if every $\mathcal{R}$-class (equivalently every $\mathcal{L}$-class) contains an idempotent.

**Theorem**

Let $M$ be a regular monoid with finitely many left and right ideals. Then $M$ has finite derivation type if and only if every maximal subgroup of $M$ has finite derivation type.

**Notes on proof.** We show in general how to construct a homotopy base for $M$ from homotopy bases of the maximal subgroups.
Complete rewriting systems

Theorem

Let $M$ be a regular monoid with finitely many left and right ideals. If every maximal subgroup of $M$ has a presentation by a finite complete rewriting system then so does $M$.

- The converse is still open.
- This relates to the following open problem from group theory:

**Question.** Is the property of having a finite complete rewriting system preserved when taking finite index subgroups?
The finiteness condition $\text{FP}_n$

- **Wall (1965):** introduced a (geometric) finiteness condition for groups called $\mathcal{F}_n$:
  - $\mathcal{F}_1 \equiv$ finite generation
  - $\mathcal{F}_2 \equiv$ finite presentability

- **Issue:** $\mathcal{F}_n$ not very tractable in terms of using algebraic machinery
- **Bieri (1976):** introduced $\text{FP}_n$ for groups.

**Definition**

A monoid $M$ is of type **left-$\text{FP}_n$** if there is a resolution:

$$F_n \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1 \twoheadrightarrow F_0 \twoheadrightarrow \mathbb{Z} \twoheadrightarrow 0$$

of the trivial left $\mathbb{Z}M$-module $\mathbb{Z}$ such that $F_0, F_1, \ldots, F_n$ are finitely generated free left $\mathbb{Z}M$-modules. A monoid is of type **left-$\text{FP}_\infty$** if it is **left-$\text{FP}_n$** for all $n \in \mathbb{N}$. 
**FP}_n \text{ and FDT**

- **Kobayashi (1990):**
  \[ M \text{ presented by a finite complete rewriting system} \rightarrow M \text{ is of type left-FP}_\infty \]

  \[ \text{FDT} \rightarrow \text{FP}_3. \]

- **Cremanns & Otto (1996):** for finitely presented groups
  \[ \text{FDT} \equiv \text{FP}_3. \]

**Corollary (of our FDT results)**

*Let M be a finitely presented regular monoid with finitely many left and right ideals. If every maximal subgroup of M is of type FP}_3 then M is of type left-FP}_3.*
A semigroup is **simple** if it has no proper ideals.

**Theorem**

*Let $S$ be a simple semigroup with finitely many left and right ideals. Then the monoid $S^1$ is of type left-$FP_n$ if and only if all of its maximal subgroups are of type $FP_n$.***

(Of course, all the maximal subgroups are isomorphic here.)
FP\textsubscript{n} for monoids with zero

**Proposition (Kobayashi (preprint))**

*If a monoid M has a zero element then M is if type left-FP\textsubscript{∞}*

**Example**

\(G\) - any group, \(M = G^0\) - adjoin a zero \((0g = g0 = 00 = 0)\).

- Maximal subgroups of \(M\) are: \(H_1 = G\), and \(H_0 = \{0\}\).
- Kobayashi \(\Rightarrow M\) is left-FP\textsubscript{∞}.
- \(G\) can have any properties we like
  - e.g. can choose \(G\) not to be of type FP\textsubscript{n} for any given \(n\).
Theorem

Let $M$ be a monoid that has a minimal ideal $G$ which is a group. Then $M$ is of type left-$FP_n$ if and only if $G$ is of type $FP_n$.

Definition

Clifford monoid - a regular monoid whose idempotents are central

Theorem

A Clifford monoid is of type left-$FP_n$ if and only if it has a minimal ideal $G$ (which is necessarily a group) and $G$ is of type $FP_n$. 
Combining the two results

- For $FP_1$ we have:

**Theorem**

Let $S$ be a monoid with a minimal ideal $J$ such that $J$ has finitely many left and right ideals. Let $G$ be a maximal subgroup of $J$. Then $S$ is of type left-$FP_1$ if and only if $G$ is of type $FP_1$.

**Corollary**

Let $S$ be a monoid with finitely many left and right ideals. Let $G$ be a maximal subgroup of the unique minimal ideal of $S$. Then $S$ is of type left-$FP_1$ if and only if $G$ is of type $FP_1$.

- Currently in the process of extending this to left-$FP_n$ ($n \geq 2$).
- **For the future:** What about monoids without minimal ideals?