Locally-finite connected-homogeneous digraphs

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joint work with R Möller (University of Iceland)

Groups and infinite graphs
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Symmetry properties for graphs

- There are varying amounts of symmetry that a graph can display.
- Roughly speaking, the more symmetry a graph has the larger its automorphism group will be (and vice versa).

Examples

\( \Gamma \) - graph, \( V\Gamma \) - vertex set

- \( \Gamma \) is **vertex-transitive** if \( \text{Aut} \ \Gamma \) acts transitively on \( V\Gamma \).
  - Cayley graphs of groups are vertex transitive.
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**Examples**

$\Gamma$ - graph, $V_{\Gamma}$ - vertex set

- $\Gamma$ is **vertex-transitive** if $\text{Aut} \, \Gamma$ acts transitively on $V_{\Gamma}$.
  - Cayley graphs of groups are vertex transitive.
- Other stronger conditions have been considered:
  - edge-transitive, arc-transitive, $k$-arc-transitive ([Tutte (1947)](Tutte1947))
  - distance-transitive ([Biggs and Smith (1971)](Biggs1971))
  - homogeneous, $k$-homogeneous ([Fraïssé (1953)](Fraisse1953))
- Concepts like this arise naturally in the theory of permutation groups.
Classification problems

- $\mathcal{P}$ - a symmetry property of graphs

**Problem**

Classify those graphs $\Gamma$ satisfying property $\mathcal{P}$.

- Various restrictions can be placed on $\Gamma$
  - e.g. we may suppose that $\Gamma$ is:
    - finite
    - infinite but locally-finite
    - countably infinite
    - arbitrary

- In the infinite locally-finite case the number of ends that the graph has plays an important role.
Ends of a graphs

Definition
The number of ends of a graph is the least upper bound (possibly $\infty$) of the number of infinite connected components that can be obtained by removing finitely many edges.

- Intuitively the number of ends corresponds to the number of “ways of going to infinity”.

Theorem (Diestel, Jung, Möller (1993))

A connected vertex-transitive graph has either 1, 2 or $\infty$ many ends.
Examples: A grid, a tree and a line

Grid has 1 end

Tree has $\infty$ many ends

Line has 2 ends
Cutting up graphs

**Definition (Cuts)**

A set $c \subseteq V\Gamma$ of vertices is called a cut if $c$ and its complement $c^*$ are both infinite and

$$\delta c = \{e \in E\Gamma : \text{one vertex of } e \text{ lies in } c \text{ and one in } c^*\}$$

is finite.

**Theorem (Dunwoody (1982))**

Any infinite connected graph with more than one end has a cut $d \subseteq V\Gamma$ such that for all $g \in \text{Aut}\Gamma$ at least one of the following holds

$$d \subseteq gd, \quad d \subseteq gd^*, \quad d^* \subseteq gd, \quad \text{or} \quad d^* \subseteq gd^*.$$
Applications of Dunwoody’s theorem

- Dunwoody’s theorem has been usefully applied in the study of locally-finite graphs satisfying symmetry conditions.

Examples

- **Macpherson (1982)** - classification of infinite locally-finite distance-transitive graphs
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Examples

- **Macpherson (1982)** - classification of infinite locally-finite distance-transitive graphs

Let $\Gamma$ be a locally finite connected graph with more than one end.

- **Möller (1992)** - If $\Gamma$ is 2-distance transitive then $\Gamma$ is $k$-distance transitive for all $k \in \mathbb{N}$.

- **Thomassen–Woess (1993)** - If $\Gamma$ is 2-arc transitive then $\Gamma$ is a regular tree.

- **Thomassen–Woess (1993)** - If $\Gamma$ is 1-arc transitive and all vertices have degree $r$, where $r$ is a prime, then $\Gamma$ is a regular tree.
Digraphs with symmetry

$D$ - a digraph, $ED \subseteq VD \times VD$ - set of arcs of $D$
(no loops or two-directional arcs $\leftrightarrow$)

**Definition**

Number of ends of $D :=$ the number of ends of the underlying undirected graph of $D$.

- **Seifter (2007)** - investigated the structure of infinite locally-finite transitive digraphs with $> 1$ end
  - They are far less “sensitive” to symmetry conditions than undirected graphs.
  - Even with a seemingly very strong condition called high-arc-transitivity they can have very rich structure.
Definition
A digraph $D$ is called connected-homogeneous if any isomorphism between finite connected induced subdigraphs of $D$ extends to an automorphism.

Example. $D =$ infinite directed line (i.e. $\mathbb{Z}$ with arcs $i \rightarrow i + 1$)
**Connected-homogeneity**

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**Example.** \( D = \) infinite directed line (i.e. \( \mathbb{Z} \) with arcs \( i \rightarrow i + 1 \))

**Problem.** Classify the countable connected-homogeneous digraphs.

A solution to this problem would complete the following table:

<table>
<thead>
<tr>
<th></th>
<th>Homogeneous</th>
<th>Connected-homogeneous</th>
</tr>
</thead>
</table>

**Subproblem.** Classify the connected-homogeneous digraphs that are **locally-finite** and have **more than one end**.
The case that a triangle embeds

**Theorem (RG & Möller (2008))**

Let $D$ be a connected locally-finite digraph with more than one end, and suppose that $D$ embeds a triangle. Then $D$ is connected-homogeneous if and only if it is isomorphic to a digraph built from directed triangles in the following way:
**Highly arc-transitive digraphs**

**Definition**

A $k$-arc in $D$ is a sequence $(x_0, \ldots, x_k)$ of vertices with $x_i \rightarrow x_{i+1}$ (and $x_{i-1} \neq x_{i+1}$).

A digraph $D$ is **highly-arc-transitive** if $\text{Aut } D$ is transitive on the set of $k$-arcs of $D$ for every natural number $k$.

- **Cameron, Praeger, and Wormald (1993)** - carried out an extensive study of the class of highly-arc-transitive digraphs.

**Proposition (RG & Möller (2008))**

Let $D$ be a triangle-free locally-finite digraph with more than one end. If $D$ is connected-homogeneous then $D$ is highly-arc-transitive.
Triangle-free connected-homogeneous digraphs

Directed regular trees
Triangle-free connected-homogeneous digraphs

Directed regular trees
Triangle-free connected-homogeneous digraphs

Other tree-like examples exist.

Constructed by gluing together certain bipartite graphs.
Definition

The set of descendants $\text{desc}(u)$ of a vertex $u$ is the set of all vertices $v$ such that there is a directed path from $u$ to $v$.

In this example $\text{desc}(u)$ is a tree for every vertex $u$. 

Triangle-free connected-homogeneous digraphs
Definition

The reachability digraph $\Delta(D)$ of $D$ is the subdigraph induced by the set of all arcs reachable by an alternating walk beginning from an arc.

In this example $\Delta(D)$ is bipartite and is isomorphic to a 6-cycle.
Triangle-free case

- For arbitrary locally finite highly-arc-transitive digraphs
  - desc($u$) need not be a tree
  - it is an open question as to whether $\Delta(D)$ is bipartite

**Theorem (RG & Möller (2008))**

Let $D$ be a triangle-free locally-finite connected-homogeneous digraph with infinitely many ends. Then

- desc($u$) is a tree for all $u \in VD$
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- desc($u$) is a tree for all $u \in VD$
- $\Delta(D)$ is a bipartite graph
- Specifically, $\Delta(D)$ is isomorphic to one of:
  - infinite semiregular tree $T_{a,b}$ ($a, b \in \mathbb{N}$)
  - cycle $C_{2m}$ ($m \geq 4$)
  - complete bipartite graph $K_{m,n}$ ($m, n \in \mathbb{N}$ with $m \geq 2$ or $n \geq 2$)
  - complement of a perfect matching $CP_n$ for some $n \geq 3$ (i.e. the complete bipartite graph $K_{n,n}$ with a matching removed)

- **Proof.** Uses Dunwoody’s theorem, structure trees, and results from Seifter (2007).
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- Specifically, \( \Delta(D) \) is isomorphic to one of:
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- But do all of these potential reachability graphs actually arise in examples?
CPW’s universal covering construction.

Let $\Delta$ be one of the following:

<table>
<thead>
<tr>
<th>semiregular tree</th>
<th>complete bipartite</th>
</tr>
</thead>
<tbody>
<tr>
<td>complement of perfect matching</td>
<td>cycle</td>
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</table>

Then there exists a connected-homogeneous digraph $DL(\Delta)$ with reachability graph $\Delta$.

$DL(\Delta)$ is constructed by gluing together copies of $\Delta$ in such a way that

- any two copies of $\Delta$ intersect in at most one vertex
- the only cycles in $D$ are those that occur in the copies of $\Delta$

This construction was introduced by Cameron, Praeger, and Wormald (1993) during their study of highly-arc-transitive digraphs.
The digraph $DL(\Delta)$ where $\Delta = C_6$ is a 6-cycle.
Triangle-free case

- And “most” examples actually arise in this way.

**Theorem (RG & Möller (2008))**

Let $D$ be a connected triangle-free locally-finite connected-homogeneous digraph with infinitely many ends, and with $\Delta(D)$ not isomorphic to $K_{2,2}$ or to the complement of a perfect matching.

Then $D \cong DL(\Delta)$, the digraph obtained from the above CPW universal covering construction.

In particular, in these cases $D$ is uniquely determined by its reachability digraph $\Delta(D)$.

- This just leaves the cases that $\Delta$ is isomorphic to $K_{2,2}$ or to the complement of a perfect matching.
$\Delta \cong \text{complement of perfect matching}$

- Malnič, Marušič, Seifter, and Zgrablić (2002)
  - introduced a new family of highly-arc-transitive digraphs
  - answered an open question about homomorphisms onto $Z$

- The original construction involved gluing together cycles $C_{2m}(m \geq 3)$.

An MMSZ digraph $D$ with $\Delta(D) \cong CP_3$ complement of perfect matching.
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- Carrying out their construction with any complement of perfect matching \( CP_m(m \geq 3) \) gives a connected-homogeneous digraph.

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Carrying out their construction with any complement of perfect matching $CP_m (m \geq 3)$ gives a connected-homogeneous digraph.

A generalisation of their construction gives further examples.
Theorem (RG & Möller (2008))

Let $D$ be a connected triangle-free locally-finite connected-homogeneous digraph with more than one end.

If $\Delta(D)$ is isomorphic to the complement of a perfect matching then either

(i) $D$ is obtained from the CPW construction or

(ii) $D$ is a generalised MMSZ digraph

In particular, for any complement of perfect matching $\Delta$ there are infinitely many non-isomorphic $D$ with $\Delta(D) \cong \Delta$. 
\[ \Delta \cong K_{2,2} - \text{the problem case} \]

\[ D - \text{connected triangle-free locally-finite connected-homogeneous digraph with more than one end.} \]

Suppose \( \Delta(D) \cong K_{2,2} \)

- Known examples
  - CPW example, and
  - generalised MMSZ examples
- But there are other examples in addition to these (too complicated to go into here :-( ).
- This is the only case where the infinitely-ended classification is still incomplete.
Concluding remarks

- We have results for the 2-ended case, e.g. $\Delta(D) \cong K_{n,n}$.

Still to do

- Complete the classification by determining all examples whose reachability graph is $K_{2,2}$.

And then

- Extend the result to:
  - non-locally finite digraphs
  - one-ended digraphs

- Generalise results to locally-finite highly-arc-transitive digraphs with more than one end.
  - In particular prove that for such digraphs $\Delta(D)$ is always bipartite.