THE WEYL EXTENSION ALGEBRA OF $GL_2(\overline{\mathbb{F}}_p)$

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Contents

1. Background

1.1. **The** 2**-category** $\mathcal{T}$**.** Let $\mathcal{T}$ denote the collection of pairs (A, M) where A is a differential k-graded algebra and M is a differential k-graded A-A-bimodule.

The collection $\mathcal{T}$ in fact forms the set of objects of a 2-category: 1-morphisms between two objects (A, M) and (B, N) are given by a pair (S, ϕ_S), consisting of a differential (bi-)graded A-B-bimodule ${}_AS_B$ and a quasi-isomorphism

$$\phi_S : S \otimes_B N \to M \otimes_A S;$$

2-morphisms from (S, ϕ_S) to (T, ϕ_T) are given by homomorphisms of differential (bi-)graded A-B-bimodules $f : S \to T$ such that the diagram

$$\begin{array}{ccc} S \otimes_B N & \xrightarrow{\phi_S} & M \otimes_A S \\ \downarrow{\scriptstyle f\otimes id} & & \downarrow{\scriptstyle id\otimes f} \\ T \otimes_B N & \xrightarrow{\phi_T} & M \otimes_A ST \end{array}$$

commutes.

Definition 1. *We define a* **Rickard object** *of $\mathcal{T}$ to be an object (A, M) of $\mathcal{T}$, where ${}_AM_A \in A\text{-perf} \cap \text{perf-}A$, the natural morphism of dg algebras $A \to \mathrm{End}_A(M)$ is a quasi-isomorphism, there is a quasi-isomorphism $A \to \mathbb{H}A$, and $\mathbb{H}A$ is a finite-dimensional algebra of finite global dimension.*

For a Rickard object (A, M) of $\mathcal{T}$, write M^{-1} for $\mathrm{Hom}_A(M, A)$.

Definition 2. *We define a j-**graded object** of $\mathcal{T}$ to be an object (a, m) of $\mathcal{T}$, where $a = \bigoplus a^{jk}$ is a differential bigraded algebra, and $m = \bigoplus m^{jk}$ a differential bigraded a-a-bimodule, and $a^{j\bullet} = m^{j\bullet} = 0$ for $j < 0$.*

1.2. **The operator $\mathbb{O}$.** Let $(\mathbf{a}, \mathbf{m})$ be a j-graded object of $\mathcal{T}$. We define

$$\mathbb{O}_{\mathbf{a},\mathbf{m}} \circlearrowright \mathcal{T}$$

to be the operator given by

$$\mathbb{O}_{\mathbf{a},\mathbf{m}}(A, M) = (\mathbf{a}(A, M), \mathbf{m}(A, M)),$$

where

$$\alpha(A, M) = (\alpha^0 \otimes A) \oplus (\bigoplus_{j>0} \alpha^j \otimes M^{\otimes_A j})$$

for $\alpha \in \{\mathbf{a}, \mathbf{m}\}$. The algebra structure on $\bigoplus \mathbf{a}^{jk} \otimes_F M^{\otimes_A j}$ is the restriction of the algebra structure on the tensor product of algebras $\mathbf{a} \otimes \mathbb{T}_A(M)$, where $\mathbb{T}_A(M)$ denotes the tensor algebra of M over A. The k-grading and differential on the complex $\bigoplus \mathbf{a}^{jk} \otimes M^{\otimes_A j}$ are defined to be the total k-grading and total differential on the tensor product of complexes. The bimodule structure, grading and differential on $\bigoplus \mathbf{m}^{jk} \otimes M^{\otimes_A j}$ are defined likewise.

We remark that this extends to a 2-endofunctor of $\mathcal{T}$ (cf. [3, Lemma 9]).

Lemma 3. [4, Lemma 14] *Let $\mathbf{a}, \mathbf{b}, \boldsymbol{c}$ be a differential bigraded algebras, ${}_{\mathbf{b}}\mathbf{x}_{\mathbf{a}}$ and ${}_{\mathbf{a}}\mathbf{y}_{\boldsymbol{c}}$ differential bigraded modules, all concentrated in nonnegative j-degrees. Let (A, M) be an object of $\mathcal{T}$. Then*

$$\mathbf{x}(A, M) \otimes_{\mathbf{a}(A,M)} \mathbf{y}(A, M) \cong (\mathbf{x} \otimes_{\mathbf{a}} \mathbf{y})(A, M)$$

as differential bigraded $\mathbf{b}(A, M)$-$\boldsymbol{c}(A, M)$-bimodules.

Given a differential bigraded $\mathbf{a}$-module $\mathbf{x}$, with components in positive and negative j-degrees, we define $\mathbf{x}(A, M)$ to be the $\mathbf{a}(A, M)$-module given by

$$\mathbf{x}(A, M) = (\bigoplus_{j<0} x^{j\bullet} \otimes (M^{-1})^{\otimes_A -j}) \oplus (x^{0\bullet} \otimes A) \oplus (\bigoplus_{j>0} x^{j\bullet} \otimes M^{\otimes_A j}),$$

where $M^{-1} := \mathrm{Hom}_A(M, A)$.

Lemma 4. *(cf.[4, Lemma 15]) Let $\boldsymbol{c}$ be a differential bigraded algebra, $\mathbf{x}$ and $\mathbf{y}$ are differential bigraded $\boldsymbol{c}$-modules, all concentrated in nonnegative j-degrees, and let (A, M) be a Rickard object of $\mathcal{T}$. Then we have a quasi-isomorphism of differential bigraded $(\boldsymbol{c}^0 \otimes A) \otimes (\boldsymbol{c}^0 \otimes A)^{op}$-modules*

$$\mathrm{Hom}_{\boldsymbol{c}}(\mathbf{x}, \mathbf{y})(A, M) \to \mathrm{Hom}_{\boldsymbol{c}(A,M)}(\mathbf{x}(A, M), \mathbf{y}(A, M)).$$

Proof. We have previously stated this only as a quasi-isomorphism of differential bigraded vector spaces. However, the quasi-isomorphism we constructed in [3, Proof of Theorem 13] is in fact a quasi-isomorphism of $(\boldsymbol{c}^0 \otimes A) \otimes (\boldsymbol{c}^0 \otimes A)^{op}$-modules. $\square$

1.3. **The operator $\mathfrak{O}$.** We now recall the definition of the operator $\mathfrak{O}$ from [4]. Let $\Gamma = \bigoplus \Gamma^{ijk}$ be a differential trigraded algebra. We have an operator

$$\mathfrak{O}_\Gamma \circlearrowleft \{\Sigma |\ \Sigma = \bigoplus \Sigma^{jk} \text{ a differential bigraded algebra } \}$$

given by

$$\mathfrak{O}_\Gamma(\Sigma)^{ik} = \bigoplus_{j,k_1+k_2=k} \Gamma^{ijk_1} \otimes \Sigma^{jk_2}. \tag{1}$$

The algebra structure and differential are obtained by restricting the algebra structure and differential from $\Gamma \otimes \Sigma$. If we forget the differential and the k-grading, the operator $\mathfrak{O}_\Gamma$ is identical to the operator $\mathfrak{O}_\Gamma$ defined in the introduction.

1.4. **Comparing $\mathbb{O}$ and $\mathfrak{O}$.** Throughout this section, let $(\mathbf{a}, \mathbf{m})$ be a j-graded object of $\mathcal{T}$ and $(A, M) \in \mathcal{T}$. Note that the algebra $\mathbb{T}_\mathbf{a}(\mathbf{m})(A, M)$, formed with respect to the j-grading on $\mathbb{T}_\mathbf{a}(\mathbf{m})$, is a differential bigraded algebra, with

$$\mathbb{T}_\mathbf{a}(m)(A, M)^{ik} = \oplus_j \mathbb{T}_\mathbf{a}(\mathbf{m})^{ijk} \otimes M^{\otimes_A j}.$$

The algebra $\mathbb{T}_{\mathbf{a}(A,M)}(\mathbf{m}(A, M))$ is a differential bigraded algebra, with

$$\mathbb{T}_{\mathbf{a}(A,M)}(\mathbf{m}(A, M))^{ik} = (\mathbf{m}(A, M)^{\otimes_{\mathbf{a}(A,M)} i})^k.$$

We write $X^{i\diamond\bullet} \cong Y^{i\diamond\bullet}$ to signify that $X^{ijk} \cong Y^{ijk}$ for all j, k.

We have the following lemmas.

Lemma 5. [4, Lemma 19]

(i) *We have an isomorphism of objects of $\mathcal{T}$*

$$\mathbb{O}_{\mathbf{a},\mathbf{m}}(A, M) \cong (\mathfrak{O}_{\mathbb{T}_\mathbf{a}(\mathbf{m})}(\mathbb{T}_A(M))^{0\diamond\bullet}, \mathfrak{O}_{\mathbb{T}_\mathbf{a}(\mathbf{m})}(\mathbb{T}_A(M))^{1\diamond\bullet}),$$

where the k-grading on the components of $\mathbb{O}_{\mathbf{a},m}(A, M)$ can be identified with the k-grading on $\mathfrak{O}_{\mathbb{T}_\mathbf{a}(\mathbf{m})}(\mathbb{T}_A(M))$.

(ii) *We have an isomorphism of differential bigraded algebras*

$$\mathfrak{O}_{\mathbb{T}_\mathbf{a}(\mathbf{m})}(\mathbb{T}_A(M)) \cong \mathbb{T}_\mathbf{a}(\mathbf{m})(A, M).$$

(iii) *We have an isomorphism of differential bigraded algebras*

$$\mathbb{T}_\mathbf{a}(\mathbf{m})(A, M) \cong \mathbb{T}_{\mathbf{a}(A,M)}(\mathbf{m}(A, M)).$$

Suppose we are given $(\mathbf{a}_i, \mathbf{m}_i)$ for $1 \le i \le n$, and (A, M). Let us define (A_i, M_i) recursively via $(A_i, M_i) = \mathbb{O}_{\mathbf{a}_i,\mathbf{m}_i}(A_{i-1}, M_{i-1})$ and $(A_0, M_0) = (A, M)$.

Lemma 6. [4, Lemma 20]

(i) *We have an algebra isomorphism*

$$\mathbb{T}_{A_n}(M_n) \cong \mathfrak{O}_{\mathbb{T}_{\mathbf{a}_n}(\mathbf{m}_n)} ... \mathfrak{O}_{\mathbb{T}_{\mathbf{a}_1}(\mathbf{m}_1)}(\mathbb{T}_A(M)).$$

(ii) *We have an isomorphism of objects of $\mathcal{T}$*

$$\mathbb{O}_{\mathbf{a}_n,\mathbf{m}_n} \cdots \mathbb{O}_{\mathbf{a}_1,\mathbf{m}_1}(A, M)$$
$$\cong (\mathfrak{O}_{\mathbb{T}_{\mathbf{a}_1}(\mathbf{m}_1)} ... \mathfrak{O}_{\mathbb{T}_{\mathbf{a}_n}(\mathbf{m}_n)}(\mathbb{T}_A(M))^{0\diamond\bullet}, \mathfrak{O}_{\mathbb{T}_{\mathbf{a}_1}(\mathbf{m}_1)} ... \mathfrak{O}_{\mathbb{T}_{\mathbf{a}_n}(\mathbf{m}_n)}(\mathbb{T}_A(M))^{1\diamond\bullet}).$$

Corollary 7. [4, Corollary 21] *Let $\mathbf{a}$ be a dg algebra and $\mathbf{m}$ dg $\mathbf{a}$-$\mathbf{a}$-bimodule. Then we have an isomorphism of algebras*

$$\mathbb{O}_{F,0}\mathbb{O}^n_{\mathbf{a},\mathbf{m}}(F, F) \cong \mathfrak{O}_F \mathfrak{O}^n_{\mathbb{T}_\mathbf{a}(\mathbf{m})}(F[z, z^{-1}]).$$

1.5. **Keller duality.** From now on, we always assume that our Rickard object (A, M) has an additional vector space d-grading, such that the d-degree 0 part of A generates the derived category $D(A)$. Let P_A be a projective resolution of the d-degree 0 part of A and let $\mathcal{K}(A)$ denote the dg algebra $\mathcal{K}(A) = \mathrm{End}_A(P_A)$. There are mutually inverse equivalences

$$D_{dg}(A) \underset{P_A \otimes_{\mathcal{K}(A)} -}{\overset{\mathrm{Hom}_A(P_A, -)}{\rightleftarrows}} D_{dg}(\mathcal{K}(A)) ,$$

by a theorem of Keller [1, Theorem 3.10]. Since P_A is projective as an A-module, we have a natural isomorphism of functors

$$\mathrm{Hom}_A(P, -) \cong \mathrm{Hom}_A(P_A, A) \otimes_A -.$$

Given a differential graded A-A-bimodule N, denote by $\mathcal{K}(M)$ the dg $\mathcal{K}(A)$-$\mathcal{K}(A)$-bimodule

$$\mathcal{K}(N) = \mathrm{Hom}_A(P_A, A) \otimes_A N \otimes_A P_A \cong \mathrm{Hom}_A(P_A, N \otimes_A P_A).$$

Lemma 8. *We have isomorphisms*

(1) $\mathcal{K}(M^{-1}) \cong \mathcal{K}(M)^{-1}$*;*

(2) $\mathcal{K}(M^{\otimes_A i}) \cong \mathcal{K}(M)^{\otimes_{\mathcal{K}(A)} i}$ *for* $i \in \mathbb{Z}$.

Proof. For simplicity, write $P = P_A$ On the on hand, we have

$$\begin{aligned}\mathcal{K}(M^{-1}) &\cong \mathrm{Hom}_A(P, \mathrm{Hom}_A(M, A) \otimes_A P) \\ &\cong \mathrm{Hom}_A(P, \mathrm{Hom}_A(M, P)) \cong \mathrm{Hom}_A(M \otimes_A P, P)\end{aligned}$$

by definition, A-projectivity of M and adjunction. On the other hand, we have

$$\begin{aligned}\mathcal{K}(M)^{-1} &= \mathrm{Hom}_{\mathcal{K}(A)}(\mathrm{Hom}_A(P, A) \otimes_A M \otimes_A P, \mathrm{Hom}_A(P, P)) \\ &\cong \mathrm{Hom}_A(P \otimes_{\mathcal{K}(A)} \mathrm{Hom}_A(P, A) \otimes_A M \otimes_A P, P).\end{aligned}$$

Since P is a projective generator for A, we have $P \otimes_{\mathcal{K}(A)} \mathrm{Hom}_A(P, A) = P \otimes_{\mathrm{End}_A(P)} \mathrm{Hom}_A(P, A) \cong A$ and (i) follows. (ii) is proved similarly. □

Definition 9. *Suppose that* $(a, \underline{m}) = (a, (m, m^{-1}))$ *is a* j*-graded Rickard object of* $\mathcal{U}$*, such that* a *is concentrated in non-negative* d*-degrees and that a projective resolution* P *of the degree zero part of* a *is differential* djk*-trigraded. Then* $\mathcal{K}(a)$ *inherits a differential* djk*-trigrading. In this case we call* $(a, \underline{m})$ *a* dagger object *of* $\mathcal{U}$*.*

2. Corrections

2.1. **The d-grading.** While the d-grading as defined in the article is a vector space grading with the property that taking the d-degree zero part of $\mathbb{HO}^q_{\boldsymbol{c},\boldsymbol{t}}(F, F)$ gives the direct sum of standard modules for a block of polynomial representations of G with p^q simple modules (and more generally, the graded pices provide a standard filtration of the corresponding algebra), the grading defined on $\boldsymbol{t}$ is not a module grading over the d-graded algebra $\boldsymbol{c}$, and as such the d-grading on $\mathbb{HO}^q_{\boldsymbol{c},\boldsymbol{t}}(F, F)$ is not an algebra grading. An algebra grading is not needed anywhere in the article, since the iterative construction only carries around the d-grading as an adornment, and can and should be phrased with the d-grading being a vector space grading only (apart from in Definition 9), with the only assumption being that the differential is of d-degree 0.

2.2. **Bonding.**

Lemma 10. *The natural maps, called bonding maps,* $M \otimes_A \mathrm{Hom}_A(M, A) \to A$ *(given by evaluation) and* $\mathrm{Hom}_A(M, A) \otimes_A M \to \mathrm{End}_A(M) \overset{qim}{\leftarrow} A$ *induce a structure of associative algebra on*

$$\mathbb{HT}_A(\underline{M}) = \mathbb{H}\left((\bigoplus_{i\geq 1} M^{\otimes_A i}) \oplus A \oplus (\bigoplus_{i\leq -1} (M'^{-1})^{\otimes_A - i}) \right).$$

Proof. Setting $E := \mathrm{End}_A(M)$, we have a commutative diagram

$$\begin{array}{ccccc}
M \otimes_A M^{-1} \otimes_A M & \xrightarrow{\phi} & M \otimes_A E & \xleftarrow{\eta} & M \otimes_A A \\
\downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \theta} & & \| \\
A \otimes_A M & & M \otimes_E E & & \\
\| & & \| & & \\
M & = & M & = & M
\end{array}$$

where ϕ, ψ are the bonding maps, $\eta : m \otimes 1 \mapsto m \otimes 1$ and $\theta : m \otimes e \mapsto m \otimes e$ are the natural maps induced by the quasi-isomorphism $A \to E$, and the equalities denote the canonical isomorphisms. Indeed in the square on the left $x \otimes f \otimes y$ maps to $f(x)y$ either way, and in the right square, $m \otimes 1$ gets sent to m either way. In homology, η and θ become isomorphisms, and $\mathbb{H}(\eta)^{-1} \circ \phi$ produces a map that makes multiplication associative.

Similarly, the commutative diagram

$$\begin{array}{ccccc}
M^{-1} \otimes_A M \otimes_A M^{-1} & \xrightarrow{\psi} & E \otimes_A M^{-1} & \xleftarrow{\eta} & A \otimes_A M^{-1} \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \theta} & & \| \\
M^{-1} \otimes_A A & & E \otimes_E M^{-1} & & \\
\| & & \| & & \\
M^{-1} & = & M^{-1} & = & M^{-1}
\end{array}$$

gives the desired associativity in homology.

□

2.3. **Iteration.**

For a Rickard object (A, M), define

$$\mathbb{KT}_A(M) := \bigoplus_{i\in\mathbb{Z}} \mathcal{K}(M^{\otimes_A i}).$$

Note that this is a priori only a $\mathcal{K}(A)$-$\mathcal{K}(A)$ bimodule but, after taking homology we obtain

$$
\begin{aligned}
\mathbb{HKT}_A(M) &= \mathbb{H}\bigoplus_{i\in\mathbb{Z}} \mathcal{K}(M^{\otimes_A i}) \\
&\cong \mathbb{H}\bigoplus_{i\in\mathbb{Z}} \mathcal{K}(M)^{\otimes_{\mathcal{K}(A)} i} \qquad (2)\\
&\cong \mathbb{HT}_{\mathcal{K}(A)}(\mathcal{K}(\underline{M}))
\end{aligned}
$$

which is an associative algebra thanks to Lemmas 10 and 8, with the algebra structure induced by the bonding maps on $\mathbb{HT}_A(M)$.

Theorem 11. *Assume the following:*

(i) Let $(\mathbf{a},\mathbf{m})$ be a dagger object of $\mathcal{T}$ such that $\mathbf{a}$ is concentrated in non-negative j-degrees.
(ii) Let $(A,M)=(A,M)$ be a Rickard object of $\mathcal{T}$, such that
  (a) both A and M are concentrated in non-negative d-degrees;
  (b) $M^{\otimes j}\otimes_A A^{0\bullet}\cong (M^{\otimes j})^{0\bullet}$ for all j such that $a^{0j\bullet}\neq 0$, where $()^{0\bullet}$ refers to taking the d-degree 0 part.
(iii) $(\mathbf{a}(A,M))^{0\bullet}$ generates the derived category $D_{dg}(\mathbf{a}(A,M))$.

Then

$$\mathbb{HKT}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M))\cong \mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}\mathbb{HT}_{\mathcal{K}(A)}(\mathcal{K}(\underline{M}))$$

Proof. Since

$$\mathbb{HKT}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M))=\mathbb{H}\bigoplus_{i\in\mathbb{Z}}\mathcal{K}(\mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}),$$

we do this for each i separately. We can choose a projective resolution of the d-degree 0 part of $\mathbf{a}(A,M)$ as $P_{\mathbf{a}}(A,M)\otimes_A P_A$ (see paper).

$$
\begin{aligned}
&\mathcal{K}(\mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i})\\
&\quad=\mathrm{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M)\otimes_A P_A, \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)\otimes_A P_A)\\
&\quad=\mathrm{Hom}_A(P_A, \mathrm{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)\otimes_A P_A))\\
&\quad=\mathrm{Hom}_A(P_A, \mathrm{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M))\otimes_A P_A)
\end{aligned}
$$

We now consider

$$\mathrm{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)).$$

If $i\geq 0$, then $\mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)\cong(\mathbf{m}^{\otimes_{\mathbf{a}} i}\otimes_{\mathbf{a}} P_{\mathbf{a}})(A,M)$ by Lemma 3, so by Lemma 4, we obtain a quasi-isomorphism

$$
\begin{aligned}
&\mathrm{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i}\otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M))\\
&\qquad\leftarrow \mathrm{Hom}_{\mathbf{a}}(P_{\mathbf{a}}, \mathbf{m}^{\otimes_{\mathbf{a}} i}\otimes_{\mathbf{a}} P_{\mathbf{a}})(A,M),
\end{aligned}
$$

and hence

$$\begin{aligned}
&\mathbb{H}\operatorname{Hom}_A(P_A, \operatorname{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i} \otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)) \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\operatorname{Hom}_A(P_A, \operatorname{Hom}_{\mathbf{a}}(P_{\mathbf{a}}, \mathbf{m}^{\otimes_{\mathbf{a}} i} \otimes_{\mathbf{a}} P_{\mathbf{a}})(A,M) \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\operatorname{Hom}_A(P_A, \mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})(A,M) \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_A(P_A, \mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})^j \otimes M^{\otimes_A j} \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\bigoplus_{j\in\mathbb{Z}} \mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})^j \otimes \operatorname{Hom}_A(P_A, M^{\otimes_A j} \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\bigoplus_{j\in\mathbb{Z}} \mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})^j \otimes \mathcal{K}(M^{\otimes_A j}) \\
&\qquad \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{H}\mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})^j \otimes \mathbb{H}(\mathcal{K}(M)^{\otimes_{\mathcal{K}(A)} j}) \\
&\qquad = \mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(m),\mathcal{K}(m^{-1}))}\mathbb{HT}_{\mathcal{K}(A)}(\mathcal{K}(A), \mathcal{K}(\underline{M}))^{i\bullet\diamond}
\end{aligned}$$

For $i < 0$

$$\begin{aligned}
&\operatorname{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i} \otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)) \\
&\qquad = \operatorname{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \operatorname{Hom}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} -i}, \mathbf{a}(A,M)) \otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)) \\
&\qquad \cong \operatorname{Hom}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} -i} \otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M), P_{\mathbf{a}}(A,M)) \\
&\qquad \cong \operatorname{Hom}_{\mathbf{a}(A,M)}((\mathbf{m}^{\otimes_{\mathbf{a}} -i} \otimes_{\mathbf{a}} P_{\mathbf{a}})(A,M), P_{\mathbf{a}}(A,M)) \\
&\qquad \leftarrow^{qim} \operatorname{Hom}_{\mathbf{a}}(\mathbf{m}^{\otimes_{\mathbf{a}} -i} \otimes_{\mathbf{a}} P_{\mathbf{a}}, P_{\mathbf{a}})(A,M),
\end{aligned}$$

so we obtain

$$\begin{aligned}
&\mathbb{H}\operatorname{Hom}_A(P_A, \operatorname{Hom}_{\mathbf{a}(A,M)}(P_{\mathbf{a}}(A,M), \mathbf{m}(A,M)^{\otimes_{\mathbf{a}(A,M)} i} \otimes_{\mathbf{a}(A,M)} P_{\mathbf{a}}(A,M)) \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\operatorname{Hom}_A(P_A, \operatorname{Hom}_{\mathbf{a}}(\mathbf{m}^{\otimes_{\mathbf{a}} -i} \otimes_{\mathbf{a}} P_{\mathbf{a}}, P_{\mathbf{a}})(A,M) \otimes_A P_A) \\
&\qquad \cong \mathbb{H}\operatorname{Hom}_A(P_A, \mathcal{K}(\mathbf{m}^{\otimes_{\mathbf{a}} i})(A,M) \otimes_A P_A)
\end{aligned}$$

from where the proof continues as in the $i > 0$ case.

The fact that this is an isomorphism of algebras follows from the algebra structure on both sides being naturally induced by the bonding maps on $(\mathbf{m}, \mathbf{m}^{-1})$ and (M, M^{-1}). Indeed, the algebra structure on $\mathbb{HKT}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M))$ is induced, thanks to Lemma 8, by that on $\mathbb{HT}_{\mathbf{a}(A,M)}(\mathbf{m}(A,M))$ which is induced by viewing it as a subalgebra inside $\mathbb{HT}_{\mathbf{a}}(\mathbf{m}) \otimes \mathbb{HT}_A(M)$. On the other hand, $\mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}\mathbb{HT}_{\mathcal{K}(A)}(\mathcal{K}(A), \mathcal{K}(\underline{M}))$ inherits its algebra structure by viewing it inside $\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m})) \otimes \mathbb{HT}_{\mathcal{K}(A)}(\mathcal{K}(\underline{M}))$ and the algebra structure of the latter is, again by Lemma 8, induced by that on $\mathbb{HT}_{\mathbf{a}}(\mathbf{m}) \otimes \mathbb{HT}_A(M)$. □

This replaces Theorem 18 in the paper, and only this statement is applied in the proof of Proposition 28, which in general form can be phrased as follows.

Corollary 12. *Let* $(\mathbf{a},\mathbf{m})$ *is a dagger object in* $\mathcal{T}$ *and* $(A,M)=\mathbb{O}^q_{\mathbf{a},\mathbf{m}}(F,F)$*. Setting* $(A_k,M_k)=\mathbb{O}^k_{\mathbf{a},\mathbf{m}}(F,F)$ *for* $k=1,\dots q$*, assume that the hypotheses of Theorem 11 are satisfied for* $(\mathbf{a},\mathbf{m})$ *and all* (A_k,M_k)*. Then*

$$\mathbb{HKT}_A(M)\cong\mathfrak{O}^q_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}F[z,z^{-1}].$$

Moreover, this yields an isomorphism

$$\mathbb{H}\mathcal{K}(A)\cong\mathfrak{O}_F\mathfrak{O}^q_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}F[z,z^{-1}].$$

Proof. We prove this by induction: in case $q=0$ both sides equal $F[z,z^{-1}]$. We then obtain

$$\begin{aligned}\mathbb{HKT}_A(M)&\cong\mathbb{HKT}_{\mathbf{a}(A_{q-1},M_{q-1})}(\mathbf{m}(A_{q-1},M_{q-1}))\\&\cong\mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}\mathbb{HT}_{\mathcal{K}(A_{q-1})}(\mathcal{K}(A_{q-1}),\mathcal{K}(\underline{M}_{q-1}))\quad\text{by Theorem 11}\\&\cong\mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}\mathbb{HKT}_{A_{q-1}}(M_{q-1})\quad\text{by (2)}\\&\cong\mathfrak{O}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}\mathfrak{O}^{q-1}_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}F[z,z^{-1}]\quad\text{by the inductive assumption}\\&=\mathfrak{O}^q_{\mathbb{HT}_{\mathcal{K}(\mathbf{a})}(\mathcal{K}(\underline{m}))}F[z,z^{-1}].\end{aligned}$$

The second statement follows by applying the operator $\mathfrak{O}_F$, and noticing that when applying it to $\mathbb{HKT}_A(M)$, we obtain precisely $\mathbb{H}\mathcal{K}(A)$. □

Proposition 28 in the paper is the direct application of this corollary to the algebras relevant for GL_2, where, setting $\mathbf{a}=\boldsymbol{c}$ and $\mathbf{m}=\boldsymbol{t}$, the algebra $\mathbb{H}\mathcal{K}(A)$ is the algebra that is there called $\mathbf{w}_q$.

2.4. **Indexing parameters** a **and** b**.** In Collary 39, we missed an extension as $\boldsymbol{d}$-$\boldsymbol{d}$-bimodules. of $\overline{M}^\tau$ by $\overline{\boldsymbol{d}^{0\sigma}}$. The generator $e_p\otimes e_1$ in $\overline{M}^\tau$ extends the element $\xi e_p\otimes e_1\xi\in\overline{\boldsymbol{d}^{0\sigma}}$, while only the other basis elements of $\overline{\boldsymbol{d}^{0\sigma}}$, namely $\xi e_h\otimes e_{p+1-h}\xi$ for $2\le h\le p-1$, split off as $\boldsymbol{d}$-$\boldsymbol{d}$-bimodules in homology. As a consequence, in the polytopal basis, any basis element $(p-1,0,0,a,b,2)\in\mathcal{P}_{\le 0}$ (in the portion coming from $\mathcal{P}_{\overline{\boldsymbol{d}^{0\sigma}}}$) should satisfy $a=b+1$ while the basis elements $(p-s,0,0,a,b,s+1)\in\mathcal{P}_{\le 0}$ for $s=2,\dots,p-1$ are correctly given as satisfying $a=b-1$. This does not affect the number of elements in the basis, nor the ijk-degree of the elements, but is relevant for the multiplication as stated in Theorem 53 to be correct.

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