On large cardinals and generalized Baire spaces

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Abstract

Working under large cardinal assumptions, we study the Borel-reducibility between equivalence relations modulo restrictions of the non-stationary ideal on some fixed cardinal $\kappa$. We show the consistency of $E_{\lambda^{++}}$, the relation of equivalence modulo the non-stationary ideal restricted to $S_{\lambda^{++}}$ in the space $(\lambda^{++})^{\lambda^{++}}$, being continuously reducible to $E_{\lambda^{++}}$, the relation of equivalence modulo the non-stationary ideal restricted to $S_{\lambda^{+}}$ in the space $2^{\lambda^{++}}$. Then we show the consistency of $E_{\kappa^{\text{reg}}}$, the relation of equivalence modulo the non-stationary ideal restricted to regular cardinals in the space $2^{\kappa}$, being $\Sigma_1^1$-complete. We finish by showing, for $\Pi_2^1$-indescribable $\kappa$, that the isomorphism relation between dense linear orders of cardinality $\kappa$ is $\Sigma_1^1$-complete.

1 Introduction

Throughout this article we assume that $\kappa$ is an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$. The equivalence relations modulo (restrictions of) the non-stationary ideal have provided a very useful tool, and a main focus of study, in generalized descriptive set theory. In [FHK] it was shown that the relation of equivalence modulo the non-stationary ideal is not a Borel relation, and that if $V = L$, then it is not $\Delta_1^1$. The equivalence relation modulo the non-stationary ideal restricted to a set stationary $S$, denoted $E_S^{\kappa}$ (see Definition 1.3), is useful when it comes to studying the complexity of the isomorphism relations of first order theories ($\cong_T$, see Definition 1.5). In [FHK] it was proved that, under some cardinality assumptions, $E_S^{\kappa}$ is Borel reducible to
\(\cong_T\) for every first order stable unsuperstable theory \(T\), where \(S^\kappa_{\lambda}\) is the set of \(\lambda\)-cofinal ordinals below \(\kappa\). Similar results were obtained in [FHK] for the other non-classifiable theories. This motivates the study of the Borel-reducibility properties of \(E^2_{S^\kappa}\).

**Theorem 1.1** ([FHK], Theorem 56). The following is consistent: For all stationary \(S\) and \(S'\), \(E^2_{S^\kappa}\) is Borel reducible to \(E^2_{S'^\kappa}\) if and only if \(S \subseteq S'\).

**Theorem 1.2** ([FHK], Theorem 55). The following is consistent: \(E^2_{S^\omega_2^\omega}\) is Borel reducible to \(E^2_{S^\omega_2^\omega}\).

In [HK] the authors used the Borel-reducibility properties of the equivalence relation modulo the non-stationary ideal to prove that in \(L\), all \(\Sigma^1_1\) equivalence relations are reducible to \(\cong_{\text{DLO}}\), where DLO is the theory of dense linear orderings without end points, which means that this equivalence relation is on top of the Borel-reducibility hierarchy among \(\Sigma^1_1\)-equivalence relations, i.e. it is \(\Sigma^1_1\)-complete. This result stands in contrast to the classical, countable case, \(\kappa = \omega\), for which it is known that all other isomorphism relation are reducible to \(\cong_{\text{DLO}}\) [FS89], but far from all \(\Sigma^1_1\)-equivalence relations are reducible to it; even some Borel-equivalence relations such as \(E_1\) are not reducible to any isomorphism relations in the countable case. So the question remained: is the \(\Sigma^1_1\)-completeness of \(\cong_{\text{DLO}}\) just a manifestation of the pathological behaviour of \(L\) or is it a more robust property in the generalised realm. One of the contributions of this paper is that the \(\Sigma^1_1\)-completeness of \(\cong_{\text{DLO}}\) is indeed a rather robust phenomenon and holds whenever \(\kappa\) has certain large cardinal properties (Theorem 3.9).

It was asked in [FHK14] and in [KLLS, Question 3.46] whether or not the equivalence relation modulo the non-stationary ideal on the Baire space can be reduced to the Cantor space for some fixed cofinality: in our notation, whether or not \(E^\kappa_{S^\mu^\kappa} \leq E^2_{S^\mu^\kappa}\). We approach the problem by proving several results in this direction. Our results have the forms

\[
E^\kappa_{S^\mu^\kappa} \leq E^2_{S^\mu^\kappa},
\]

\[
E^\kappa_{S^\mu^\kappa} \leq E^2_{\text{reg}(\kappa)},
\]

and

\[
E^\kappa_{\text{reg}(\kappa)} \leq E^2_{\text{reg}(\kappa)}.
\]
where $\mu^*$ is larger than $\mu$ and $\text{reg}(\kappa)$ is the set of regular cardinals below $\kappa$ Mahlo. These results are obtained under various assumptions and sometimes in forcing extensions.

Many of the results in the area of reducibility of equivalence relations modulo non-stationary ideals use combinatorial principles, like $\diamondsuit$, and other reflection principles. In this paper we bring also some large cardinal principles into the picture.

The generalized Baire space is the set $\kappa^\kappa$ with the bounded topology. For every $\zeta \in \kappa^{<\kappa}$, the set \[
[\zeta] = \{ \eta \in \kappa^\kappa \mid \zeta \subset \eta \}\]
is a basic open set. The open sets are of the form $\bigcup X$ where $X$ is a collection of basic open sets. The collection of $\kappa$-Borel subsets of $\kappa^\kappa$ is the smallest set which contains the basic open sets and is closed under unions and intersections of length $\kappa$. Since in this paper we do not consider any other kind of Borel sets besides $\kappa$-Borel, we will omit the prefix “$\kappa$-”.

The generalized Cantor space is the subspace $2^\kappa \subset \kappa^\kappa$ with the relative subspace topology. For $X,Y \in \{\kappa^\kappa,2^\kappa\}$, we say that a function $f : X \to Y$ is Borel if for every open set $A \subseteq Y$ the inverse image $f^{-1}[A]$ is a Borel subset of $X$. Let $E_1$ and $E_2$ be equivalence relations on $X$ and $Y$ respectively. We say that $E_1$ is Borel reducible to $E_2$ if there is a Borel function $f : X \to Y$ that satisfies $(\eta,\xi) \in E_1 \iff (f(\eta),f(\xi)) \in E_2$. We call $f$ a reduction of $E_1$ to $E_2$. This is denoted by $E_1 \leq_B E_2$, and if $f$ is continuous, then we say that $E_1$ is continuously reducible to $E_2$, which is denoted by $E_1 \leq_c E_2$.

For every stationary $S \subset \kappa$, we define the equivalence relation modulo the non-stationary ideal restricted to a stationary set $S$, on the space $\lambda^\kappa$ for $\lambda \in \{2,\kappa\}$.

**Definition 1.3.** For every stationary $S \subset \kappa$ and $\lambda \in \{2,\kappa\}$, we define $E_S^{\lambda,\kappa}$ as the relation \[
E_S^{\lambda,\kappa} = \{ (\eta,\xi) \in \lambda^\kappa \times \lambda^\kappa \mid \{ \alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha) \} \cap S \text{ is not stationary} \}.
\]

Note that $E_S^{2,\kappa}$ can be identified with the equivalence relation on the power set of $\kappa$ in which two sets $A$ and $B$ are equivalent if their symmetric difference restricted to $S$ is non-stationary. This can be done by identifying a set $A \subset \kappa$ with its characteristic function.

For every regular cardinal $\mu < \kappa$, we denote $\{ \alpha < \kappa \mid \text{cf}(\alpha) = \mu \}$ by $S_\mu^\kappa$. A set $C$ is $\mu$-club if it is unbounded and closed under $\mu$-limits. For brevity, when $S = S_\mu^\kappa,$
we will denote $E_{\lambda,\kappa}^\mu$ by $E_{\mu\text{-club}}$. Note that $(f, g) \in E_{\mu\text{-club}}^\lambda,\kappa$ if and only if the set \( \{ \alpha < \kappa \mid f(\alpha) = g(\alpha) \} \) contains a \( \mu \)-club.

For a Mahlo cardinal \( \kappa \), the set \( \text{reg}(\kappa) = \{ \alpha < \kappa \mid \alpha \text{ a regular cardinal} \} \) is stationary. We will denote the equivalence relation \( E_{\text{reg}(\kappa)}^{\lambda,\kappa} \) by \( E_{\text{reg}}^{\lambda,\kappa} \).

Given an equivalence relation \( E \) on \( X \in \{ \kappa, 2^\kappa \} \), we can define the \( \lambda \)-product relation of \( E \) for any \( 0 < \lambda < \kappa \). The \( \lambda \)-product relation \( \Pi^\lambda_E \) is the relation defined on \( X^\lambda \times X^\lambda \) by \( \eta \Pi^\lambda_E \xi \) if \( \eta \gamma \Pi E \xi \gamma \) holds for every \( \gamma < \lambda \), where \( \eta = (\eta_\gamma)_{\gamma < \lambda} \) and \( \xi = (\xi_\gamma)_{\gamma < \lambda} \). We endow the space \( X^\lambda, X \in \{ \kappa, 2^\kappa \} \), with the box topology generated by the basic open sets:

\[
\{ \Pi^\lambda_{\alpha < \lambda} O_\alpha \mid \forall \alpha < \lambda (O_\alpha \text{ is an open set in } X) \}.
\]

One of the motivations to study Borel reducibility in generalized Baire spaces is the connection with model theory. This connection consists in the possibility to study the Borel reducibility of the isomorphism relation of theories by coding structures with universe \( \kappa \) via elements of \( \kappa^\kappa \). We may fix this coding, relative to a given countable relational vocabulary \( L = \{ P_n \mid n < \omega \} \), as in the following definition.

**Definition 1.4.** Fix a bijection \( \pi : \kappa^{<\omega} \to \kappa \). For every \( \eta \in \kappa^\kappa \) define the \( L \)-structure \( A_\eta \) with universe \( \kappa \) as follows: For every relation \( P_m \) with arity \( n \), every tuple \( (a_1, a_2, \ldots, a_n) \) in \( \kappa^n \) satisfies

\[
(a_1, a_2, \ldots, a_n) \in P^A_m \iff \eta(\pi(m, a_1, a_2, \ldots, a_n)) \geq 1.
\]

When we describe a complete theory \( T \) in a vocabulary \( L' \subseteq L \), we think of it as a complete \( L \)-theory extending \( T \cup \{ \forall \bar{x} \rightarrow P_n(\bar{x}) \mid P_n \in L \setminus L' \} \).

**Definition 1.5 (The isomorphism relation).** Assume \( T \) is a complete first order theory in a countable vocabulary. We define \( \cong_T \) as the relation

\[
\{ (\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid (A_\eta \models T, A_\xi \models T, A_\eta \cong A_\xi) \text{ or } (A_\eta \not\models T, A_\xi \not\models T) \}.
\]

In the second section we will study the reducibility between different cofinalities, and in the last section we will study the reducibility of \( E_{\text{reg}}^{\kappa,\kappa} \) and \( E_{\text{reg}}^{\kappa,2^\kappa} \). Here is the list of the main results in this article:

- **(Theorem 2.11)** Suppose \( \kappa \) is a \( \Pi^1_+ \)-indescribable cardinal for some \( \lambda < \kappa \) and \( V = L \). Then there is a forcing extension where \( \kappa \) is collapsed to become \( \lambda^++ \) and \( E_{\lambda\text{-club}}^{\lambda^++,\lambda^+} \leq_c E_{\lambda^+\text{-club}}^{2^{\lambda^+},\lambda^+} \).
• (Corollary 2.14) The following statement is consistent relative to the consistency of countably many supercompact cardinals: $E_{\omega^2,\omega_2}^{\omega^2,\omega_2} \leq c E_{\omega_1,\omega_1}^{\omega^2,\omega_2}$, and for every $n > 2$ and every $0 \leq k \leq n - 3$, $E_{\omega_k,\omega_k}^{\omega_n,\omega_n} \leq c E_{\omega_{n-1},\omega_{n-1}}^{\omega_n,\omega_n}$.

This corollary follows from [[JS], Theorem 1.3] and gives a model (different from $L$ or the one in Theorem 1.2) in which reducibility between different cofinalities holds.

• (Theorem 3.3) Suppose $S = S^\kappa_\lambda$ for some regular cardinal $\lambda < \kappa$, or $S = \text{reg}(\kappa)$ and $\kappa$ weakly compact. If $\kappa$ has the $S$–dual diamond (Definition 3.2), then $E^S_{\kappa,\kappa} \leq c E^{2,\kappa}_{\text{reg}}$.

• (Corollary 3.5) Suppose $V = L$ and $\kappa$ is weakly compact. Then $E^{2,\kappa}_{\text{reg}}$ is $\Sigma_1^1$-complete.

• (Theorem 3.6) Suppose $\kappa$ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^\kappa_{\text{reg}} \leq c E^{2,\kappa}_{\text{reg}}$ and $\kappa$ is still supercompact in the extension.

• (Theorem 3.7) If $\kappa$ is a $\Pi^1_2$-indescribable cardinal, then $E^{\kappa,\kappa}_{\text{reg}}$ is $\Sigma_1^1$-complete.

• (Corollary 3.8) Suppose $\kappa$ is a supercompact cardinal. There is a generic extension $V[G]$ in which $\kappa$ is still supercompact and $E^{2,\kappa}_{\text{reg}}$ is $\Sigma_1^1$-complete.

• (Theorem 3.9) Let $\text{DLO}$ be the theory of dense linear orderings without end points. If $\kappa$ is a $\Pi^1_2$-indescribable cardinal, then $\cong_{\text{DLO}}$ is $\Sigma_1^1$-complete.

2 Reducibility between different cofinalities

In [FHK] the authors studied the reducibility between the relations $E^{2,\kappa}_{\mu,\text{club}}$ and showed in particular the consistency of $E^{2,\lambda,++}_{\lambda,\text{club}} \leq c E^{2,\lambda,++}_{\lambda,\text{club}}$. In this section we continue along these lines.

Definition 2.1. We say that a set $X \subset \kappa$ strongly reflects to a set $Y \subset \kappa$ if for all stationary $Z \subset X$ there exist stationary many $\alpha \in Y$ with $Z \cap \alpha$ stationary in $\alpha$.

In [FHK, Theorem 55] it is proved that: If $\kappa$ is a weakly compact cardinal, then $S^\kappa_\lambda$ strongly reflects to $\text{reg}(\kappa)$, for any regular cardinal $\lambda < \kappa$. This result can be generalized to $\Pi^1_1$-indescribable cardinals:

Definition 2.2. A cardinal $\kappa$ is $\Pi^1_1$-indescribable if whenever $A \subset V_\kappa$ and $\sigma$ is a $\Pi_1$ sentence such that $(V_{\kappa+\lambda}, \in, A) \models \sigma$, then for some $\alpha < \kappa$, $(V_{\alpha+\lambda}, \in, A \cap V_\alpha) \models \sigma$. 5
Strongly unfoldable cardinals are examples of $\Pi_1^\lambda$-indescribable cardinals.

**Lemma 2.3.** Suppose $\kappa$ is a $\Pi_1^\lambda$-indescribable cardinal. There are $\lambda$ many disjoint stationary subsets of $\kappa$, $\langle S_\gamma \rangle_{\gamma < \lambda}$, such that for every $\gamma < \lambda$, $S_\gamma \subseteq \text{reg}(\kappa)$ and $\kappa$ strongly reflects to $S_\gamma$.

**Proof.** Let $S_\beta^*$ denote the set of all the $\Pi_1^\beta$-indescribable cardinals below $\kappa$. Since “$\kappa$ is $\Pi_1^\beta$-indescribable” is a $\Pi_1^1$ property of the structure $(V_{\kappa+\lambda}, \in)$, the set $S_\beta^*$ is stationary for every $\beta < \lambda$.

Let us show that for every stationary set $X \subseteq \kappa$,

$$B = \{ \alpha \in S_\beta^* \mid X \cap \alpha \text{ is stationary in } \alpha \}$$

is stationary. Let $C$ be a club in $\kappa$. The sentence

$$(C \text{ is unbounded in } \kappa) \land (X \text{ is stationary in } \kappa) \land (\kappa \text{ is } \Pi_1^\beta \text{-indescribable})$$

is a $\Pi_1^1$ property of the structure $(V_{\kappa+\lambda}, \in, X, C)$. By reflection, there is $\gamma < \kappa$ such that $C \cap \gamma$ is unbounded in $\gamma$, and hence $\gamma \in C$, $S \cap \gamma$ is stationary in $\gamma$, and $\gamma$ is $\Pi_1^\beta$-indescribable. We conclude that $C \cap B \neq \emptyset$.

Let us denote $S_\beta^* \setminus S_{\beta+1}^*$ by $S_\beta$. Let us show that for every stationary set $X \subseteq \kappa$,

$$\{ \alpha \in S_\beta \mid X \cap \alpha \text{ is stationary in } \alpha \}$$

is stationary. Let $C$ be a club in $\kappa$. Since $\{ \alpha \in S_\beta^* \mid X \cap \alpha \text{ is stationary in } \alpha \}$ is stationary, we can pick $\gamma \in C \cap \{ \alpha \in S_\beta^* \mid X \cap \alpha \text{ is stationary in } \alpha \}$ such that $\gamma$ is minimal.

**Claim 2.3.1.** $\gamma$ is not $\Pi_1^{\beta+1}$-indescribable.

**Proof.** Suppose, towards a contradiction, that $\gamma$ is $\Pi_1^{\beta+1}$-indescribable. The sentence

$$(C \cap \gamma \text{ is unbounded in } \gamma) \land (X \cap \gamma \text{ is stationary in } \gamma) \land (\gamma \text{ is } \Pi_1^\beta \text{-indescribable})$$

is a $\Pi_1^1$ property of the structure $(V_{\gamma+\beta+1}, \in, X \cap \gamma, C \cap \gamma)$. By reflection, there is $\gamma' < \gamma$ such that $C \cap \gamma'$ is unbounded in $\gamma'$, $X \cap \gamma'$ is stationary in $\gamma'$, and $\gamma'$ is $\Pi_1^\beta$-indescribable. This contradicts the minimality of $\gamma$. □

We conclude that $S_\beta$ is stationary and $\{ \alpha \in S_\beta \mid X \cap \alpha \text{ is stationary in } \alpha \}$ is stationary, for every $\beta < \lambda$. □
The notion of $\diamond$-reflection was introduced in [FHK] in order to find reductions between equivalence relations modulo non-stationary ideals (see below).

**Definition 2.4** ($\diamond$-reflection). Let $X, Y$ be subsets of $\kappa$ and suppose $Y$ consists of ordinals of uncountable cofinality. We say that $X$ $\diamond$-reflects to $Y$ if there exists a sequence $(D_\alpha)_{\alpha \in Y}$ such that:

- $D_\alpha \subset \alpha$ is stationary in $\alpha$ for all $\alpha \in Y$.
- if $Z \subset X$ is stationary, then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

**Theorem 2.5** ([FHK], Theorem 59). Suppose $V = L$ and that $X \subseteq \kappa$ and $Y \subseteq \operatorname{reg}(\kappa)$. If $X$ strongly reflects to $Y$, then $X \diamond$-reflects to $Y$.

**Theorem 2.6** ([FHK], Theorem 58). If $X \diamond$-reflects to $Y$, then $E^X_X \leq c E^Y_Y$.

$\diamond$-reflection also implies some reductions for the relations $E^\kappa_\mu$ on the space $\kappa^\kappa$. To show this, we first need to introduce some definitions.

**Definition 2.7.** For every $\alpha < \kappa$ with $\gamma < \operatorname{cf}(\alpha)$ define $E^\kappa_\gamma \upharpoonright \alpha$ by:

$$E^\kappa_\gamma \upharpoonright \alpha = \{(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \mid \exists C \subseteq \alpha \gamma\text{-club}, \forall \beta \in C, \eta(\beta) = \xi(\beta)\}.$$

**Proposition 2.8.** Suppose $\gamma < \lambda < \kappa$ are regular cardinals. If $S^\kappa_\gamma$ strongly reflects to $S^\kappa_\lambda$, then $E^\kappa_\gamma \leq c E^\kappa_\lambda$.

**Proof.** Suppose that for every stationary set $S \subset S^\kappa_\gamma$ it holds that $\{\alpha \in S^\kappa_\lambda \mid S \cap \alpha$ is stationary in $\alpha\}$ is a stationary set, and define $F: \kappa^\kappa \rightarrow \kappa^\kappa$ by

$$F(\eta)(\alpha) = \begin{cases} f_\alpha(\eta), & \text{if } \operatorname{cf}(\alpha) = \lambda \\ 0, & \text{otherwise.} \end{cases}$$

where $f_\eta(\alpha)$ is a code in $\kappa \setminus \{0\}$ for the $(E^\kappa_\gamma \upharpoonright \alpha)$-equivalence class of $\eta$.

Let us prove that if $(\eta, \xi) \in E^\kappa_\gamma$, then $(F(\eta), F(\xi)) \in E^\kappa_\lambda$. Suppose $(\eta, \xi) \in E^\kappa_\gamma$. There is a $\gamma$-club where $\eta$ and $\xi$ coincide and so there is a club $C$ such that for all $\alpha \in C \cap S^\kappa_\lambda$ the functions $\eta$ and $\xi$ are $(E^\kappa_\lambda \upharpoonright \alpha)$-equivalent. Thus, by the definition of $F$, for all $\alpha \in C \cap S^\kappa_\lambda$, $F(\eta)(\alpha) = F(\xi)(\alpha)$. We conclude that $(F(\eta), F(\xi)) \in E^\kappa_\lambda$.

Let us prove that if $(\eta, \xi) \notin E^\kappa_\gamma$, then $(F(\eta), F(\xi)) \notin E^\kappa_\lambda$. Suppose that $(\eta, \xi) \notin E^\kappa_\gamma$. Then there is a stationary $S \subset S^\kappa_\gamma$ on which $\eta(\alpha) \neq \xi(\alpha)$. Since $A = \{\alpha \in S^\kappa_\lambda \mid S \cap \alpha$ is stationary in $\alpha\}$ is a stationary and for all $\alpha \in A$, $f_\alpha(\eta) \neq f_\alpha(\xi)$, we conclude that $(F(\eta), F(\xi)) \notin E^\kappa_\lambda$. \qed
Corollary 2.9. Suppose $\gamma < \lambda < \kappa$ are regular cardinals. If $S^\kappa_\gamma$ $\diamond$-reflects to $S^\kappa_\lambda$, then

(i) $E^{2,\kappa}_{\gamma}$-club $\leq c E^{2,\kappa}_{\lambda}$-club.

(ii) $E^{\kappa,\kappa}_{\gamma}$-club $\leq c E^{\kappa,\kappa}_{\lambda}$-club.

Proof. (i) Follows from Theorem 2.6.

(ii) By the definition of $\diamond$-reflection, $S^\kappa_\gamma$ $\diamond$-reflecting to $S^\kappa_\lambda$ implies that for all $S \subseteq S^\kappa_\gamma$ the set $\{ \alpha \in S^\kappa_\lambda | S \cap \alpha \text{ is stationary in } \alpha \}$ is a stationary set. The result follows from Proposition 2.8.

In [FHK], the consistency of $S^{\lambda++}_\kappa$ $\diamond$-reflecting to $S^{\lambda++}_\lambda$ was shown. This gives a model in which $E^{2,\kappa}_{\lambda}$-club $\leq c E^{2,\kappa}_{\lambda++}$ and $E^{\lambda++}_{\lambda}$-club $\leq c E^{\lambda++}_{\lambda}$-club.

Theorem 2.10 ([FHK], Theorem 55). Suppose that $\kappa$ is a weakly compact cardinal and $V = L$. Then:

(i) $E^{2,\kappa}_{\lambda}$-club $\leq c E^{2,\kappa}_{\reg}$ holds for all regular $\lambda < \kappa$.

(ii) For every regular $\lambda < \kappa$ there is a forcing extension where $\kappa$ is collapsed to become $\lambda$ and $E^{2,\lambda++}_{\lambda}$-club $\leq c E^{2,\lambda++}_{\lambda}$-club.

The proof of this theorem can be generalised using Lemma 2.3 to show the consistency of $E^{\lambda++}_{\lambda}$-club $\leq c E^{\lambda++}_{\lambda}$-club.

Theorem 2.11. Suppose $\kappa$ is a $\Pi^1_1$-indescribable cardinal and that $V = L$. Then there is a forcing extension where $\kappa$ is collapsed to become $\lambda$ and $E^{\lambda++}_{\lambda}$-club $\leq c E^{2,\lambda++}_{\lambda}$-club.

Proof. Let us collapse $\kappa$ to $\lambda$ with the Levy collapse

$$\mathbb{P} = \{ f : \reg(\kappa) \to \kappa^{<\lambda^+} | \rang(f(\mu)) \subseteq \mu, |\{\mu | f(\mu) \neq \emptyset\}| \leq \lambda \}$$

where $f \geq g$ if and only if $f(\mu) \subseteq g(\mu)$ for all $\mu \in \reg(\kappa)$. Let us define $\mathbb{P}_\mu$ and $\mathbb{P}^\mu$ for all $\mu$ by: $\mathbb{P}_\mu = \{ f \in \mathbb{P} | \sprt(f) \subseteq \mu \}$ and $\mathbb{P}^\mu = \{ f \in \mathbb{P} | \sprt(f) \subseteq \kappa \setminus \mu \}$. It is known that all regular $\lambda < \mu \leq \kappa$ satisfy:

(i) if $\mu > \lambda^+$, then $\mathbb{P}_\mu$ has the $\mu$-c.c.,
(ii) \( \mathbb{P}_{\mu} \) and \( \mathbb{P}^\mu \) are \(<\lambda^+\)-closed,

(iii) \( \mathbb{P} = \mathbb{P}_\kappa \models \lambda^+ = \check\kappa, \)

(iv) if \( \mu < \kappa \), then \( \mathbb{P} \models cf(\check\mu) = \lambda^+, \)

(v) if \( p \in \mathbb{P}, \sigma \) a name, and \( p \models \sigma \) “is a club in \( \lambda^{++} \)”, then there is a club \( E \subset \kappa \) such that \( p \models \check{E} \subset \sigma. \)

Claim 2.11.1. There is a sequence \( \langle S_\gamma \rangle_{\gamma < \lambda^+} \) of disjoint stationary subsets of \( S^{\lambda^+}_\lambda \) in \( V[G] \) such that \( S^{\lambda^+}_\lambda \) \( \diamond \)-reflects to \( S_\gamma \) for every \( \gamma < \lambda^+ \).

Proof. Let \( G \) be a \( \mathbb{P} \)-generic over \( V \), and define \( G_{\mu} = G \cap \mathbb{P}_{\mu} \) and \( G^\mu = G \cap \mathbb{P}^\mu \). So \( G_{\mu} \) is \( \mathbb{P}_{\mu} \)-generic over \( V \), \( G^\mu \) is \( \mathbb{P}^\mu \)-generic over \( V[G_{\mu}] \), and \( V[G] = V[G_{\mu}][G^\mu] \). Let \( S^\beta_\beta \) denote the set of all the \( \Pi^3_1 \)-indescribable cardinals below \( \kappa \) and \( S^\beta_\beta = S^\beta_\beta \setminus S^\beta_{\beta+1} \). We will show that \( S^{\lambda^+} \) \( \diamond \)-reflects to \( \{ \mu \in V[G] \mid \mu \in S^\gamma_\beta \} \) for all \( \beta < \lambda^+ \). Let us fix \( \beta < \lambda^+ \) and denote by \( Y \) the set \( \{ \mu \in V[G] \mid \mu \in S^\gamma_\beta \} \). By Lemma 2.3 we know that \( S^\gamma_\beta \) is stationary and by (v), it remains stationary in \( V[G] \). By (i) we know that there are no antichains of length \( \mu \) in \( \mathbb{P}_{\mu} \), and since \( |\mathbb{P}_{\mu}| = \mu \) we conclude that there are at most \( \mu \) antichains. On the other hand, there are \( \mu^+ \) many subsets of \( \mu \). Hence, there is a bijection

\[
h_{\mu} : \mu^+ \to \{ \sigma \mid \sigma \text{ is a nice } \mathbb{P}_{\mu} \text{ name for a subset of } \mu \}
\]

for each \( \mu \in \text{reg}(\kappa) \) such that \( \mu > \lambda^+ \), where a nice \( \mathbb{P}_{\mu} \) name for a subset of \( \check{\mu} \) is of the form \( \bigcup \{ \{ \check{\alpha} \} \times A_{\alpha} \mid \alpha \in B \} \) with \( B \subset \check{\mu} \) and \( A_{\alpha} \) an antichain in \( \mathbb{P}_{\mu} \). Notice that the nice \( \mathbb{P}_{\mu} \) names for subsets of \( \check{\mu} \) are subsets of \( V^\mu_\mu \). Let us define

\[
D_{\mu} = \begin{cases} [h_{\mu}(([\bigcup G])(\mu^+))(0))]_G & \text{if this set is stationary} \\ \mu & \text{otherwise.} \end{cases}
\]

We will show that \( \langle D_{\mu} \rangle_{\mu \in Y} \) is the needed \( \diamond \)-sequence in \( V[G] \).

Suppose, towards a contradiction, that there are a stationary set \( S \subset S^{\lambda^+} \) and a club \( C \subset \lambda^+ \) (in \( V[G] \)) such that for all \( \alpha \in C \cap Y \), \( D_{\alpha} \neq S \cap \alpha \). By (v) there is a club \( C_0 \subset C \) such that \( C_0 \in V \). Let \( \check{S} \) be a nice name for \( S \) and \( p \) a condition such that \( p \) forces that \( \check{S} \) is stationary. We will show that

\[
H = \{ q < p \mid q \models D_{\mu} = \check{S} \cap \check{\mu} \text{ for some } \mu \in C_0 \}
\]
is dense below \( p \), which is a contradiction. Let us redefine \( \mathbb{P} \). Let \( \mathbb{P}^* = \{ q \mid \exists r \in \mathbb{P} (r \upharpoonright \text{sprt}(r) = q) \} \). Clearly \( \mathbb{P} \cong \mathbb{P}^* \), \( \mathbb{P}^* \subseteq V_\kappa \), and \( \mathbb{P}_\mu^* = \mathbb{P}^* \cap V_\mu \), where \( \mathbb{P}_\mu^* = \{ q \mid \exists r \in \mathbb{P}_\mu (r \upharpoonright \text{sprt}(r) = q) \} \). It can be verified that the properties mentioned above also hold for \( \mathbb{P}_\mu^* \). From now on denote \( \mathbb{P}_\mu^* \) by \( \mathbb{P}_\mu \). Let \( r \) be a condition stronger than \( p \) and

\[
R = (\mathbb{P} \times \{0\}) \cup (\mathcal{S} \times \{1\}) \cup (C_0 \times \{2\}) \cup (\{r\} \times \{3\}).
\]

Let \( \forall A\varphi \) be the formula:

If \( A \) is closed and unbounded and \( t < r \) are arbitrary, then there exists \( q < r \) and \( \alpha \in A \) such that \( q \models \varphi \). Clearly, \( \forall A\varphi \) says \( r \models (\mathcal{S} \text{ is stationary}) \). By (v) it is enough to quantify over club sets in \( V \). Notice that \( t < r \), \( q < t \), \( A \) is a club, and \( \alpha \in A \) are first order expressible using \( R \) as a parameter. The definition of \( \tilde{\alpha} \) is recursive in \( \alpha \):

\[
\tilde{\alpha} = \{(\tilde{\gamma}, 1_p) \mid \gamma < \alpha\}
\]

and it is absolute for \( V_\kappa \). Then \( q \models p \tilde{\alpha}, q \in \mathcal{S} \) is equivalent to saying that for each \( q' < q \) there exists \( q'' < q' \) with \( (\tilde{\alpha}, q'') \in \mathcal{S} \), and this is first order expressible using \( R \) as a parameter. Therefore \( \forall A\varphi \) is a \( \Pi_1^1 \) property of the structure \( (V_\kappa, \in, R) \), even more

\[
(\forall A\varphi) \land (\kappa \text{ is } \Pi_1^1 \text{-indescribable})
\]

is a \( \Pi_1^1 \) property of the structure \( (V_{\kappa + \lambda}, \in, R) \). By reflection, there is \( \mu < \kappa \) \( \Pi_1^2 \)-indescribable, such that \( \mu \in C_0 \), \( r \in \mathbb{P}_\mu \), and \( (V_\mu, \in, R) \models \forall A\varphi \). In the same way as in Claim 2.3.1, we can show that there is is \( \mu < \kappa \) \( \Pi_1^2 \)-indescribable that is not \( \Pi_1^{\beta+1} \)-indescribable, i.e. \( (\mu_G \in Y)^{V[G]} \), such that \( \mu \in C_0 \), \( r \in \mathbb{P}_\mu \), and \( (V_\mu, \in, R) \models \forall A\varphi \). Notice that \( \alpha \in S \cap \mu \) implies that \( (\tilde{\alpha}, q) \in \mathcal{S} \) for some \( q \in \mathbb{P}_\mu \). Let \( \mathcal{S}_\mu = \mathcal{S} \cap V_\mu \), thus \( r \models p \mathcal{S}_\mu \) (\( \mathcal{S}_\mu \) is stationary). Let us define \( q \) as follows:

\[
\text{dom}(q) = \text{dom}(r) \cup \{\mu^+\}, q \upharpoonright \mu = r \upharpoonright \mu \text{ and } q(\mu^+) = f, \text{ dom}(f) = \{0\}, \text{ and } f(0) = h^{-1}_\mu(\mathcal{S}_\mu). \]

Since \( \mathbb{P}_\mu \) is \( \lambda^+ \)-closed and does not kill stationary subsets of \( S_\lambda^{\alpha+} \), \( (\mathcal{S}_\mu)_{G_\mu} \) is stationary in \( V[G] \), and by the way we chose \( \mu \), \( (\mathcal{S}_\mu)_{G_\mu} = (\mathcal{S}_\mu)_{G} \). Therefore \( q \models (\mathcal{S}_\mu \text{ is stationary}) \), and by the definition of \( D_\mu \) (in \( V[G] \)) we conclude that \( q \models \mathcal{S}_\mu = D_\mu \). Finally, by the way we chose \( \mu \), we get that \( (\mathcal{S}_\mu)_{G} = S \cap \mu \). We conclude that \( H \) is dense below \( p \), a contradiction.

From now on in this proof, we will work in \( V[G] \). In particular, \( \kappa \) will be \( \lambda^{++} \).

Claim 2.11.2. \( E_{\lambda, \text{club}}^{\nu, \kappa} \leq_c \Pi_1^\lambda \leq^{2, \kappa}_{\lambda, \text{club}} \).
Proof. Let $H$ be a bijection from $\kappa$ to $2^\lambda^+$. Define $F : \kappa^\kappa \to (2^\kappa)^\lambda^+$ by $F(f) = (f_\gamma)_{\gamma < \lambda^+}$, where $f_\gamma(\alpha) = H(f(\alpha))(\gamma)$ for every $\gamma < \lambda^+$ and $\alpha < \kappa$. Let us show that $F$ is a reduction of $E_{\lambda^+}^{\kappa,\kappa}$ to $\Pi_{\lambda^+}^1 E_{\lambda^+}^{2,\kappa,\kappa}$.

Clearly $f(\alpha) = g(\alpha)$ implies $H(f(\alpha)) = H(g(\alpha))$ for every $\gamma < \lambda^+$. Therefore, $f E_{\lambda^+}^{\kappa,\kappa}$ $g$ implies that for all $\gamma < \lambda^+$, $f_\gamma E_{\lambda^+}^{2,\kappa,\kappa}$ $g_\gamma$ holds. So $f \Pi_{\lambda^+}^1 E_{\lambda^+}^{1,\kappa}$ $g$.

Suppose that for every $\gamma < \lambda^+$ there is $C_\gamma$, a $\lambda$-club, such that $f_\gamma(\alpha) = g_\gamma(\alpha)$ holds for every $\alpha \in C_\gamma$. Since the intersection of less than $\kappa$ $\lambda$-club sets is a $\lambda$-club set, there is a $\lambda$-club $C$ on which the functions $f_\gamma$ and $g_\gamma$ coincide for every $\gamma < \lambda^+$. Therefore $H(f(\alpha))(\gamma) = H(g(\alpha))(\gamma)$ holds for every $\gamma < \lambda^+$ and every $\alpha \in C$, so $H(f(\alpha)) = H(g(\alpha))$ for every $\alpha \in C$. Since $H$ is a bijection, we can conclude that $f(\alpha) = g(\alpha)$ for every $\alpha \in C$, and hence $f E_{\lambda^+}^{\kappa,\kappa} g$.

By Claim 2.11.1, there is a sequence $\langle S_\gamma \rangle_{\gamma < \lambda^+}$ of disjoint stationary subsets of $S_\lambda^\kappa$ such that $S_\lambda^\kappa \circ$-reflects to $S_\gamma$ for all $\gamma < \lambda^+$. Let $\langle D_\alpha^\gamma \rangle_{\alpha \in S_\gamma}$ be a sequence that witnesses that $S_\lambda^\kappa \circ$-reflects to $S_\gamma$.

For every $\eta \in \kappa^\kappa$ define $F(\eta)$ by:

$$F(\eta)(\alpha) = \begin{cases} 1 & \text{if there is } \gamma < \lambda^+ \text{ with } \alpha \in S_\gamma \text{ and } F(\eta)_\gamma^{-1}[1] \cap D_\alpha^\gamma \text{ stationary in } \alpha \\ 0 & \text{otherwise} \end{cases}$$

where $(F(\eta)_\gamma)_{\gamma < \lambda^+} = F(\eta)$ and where $F$ is the reduction given by Claim 2.11.2.

Suppose $\eta$, $\xi$ are not $E_{\lambda^+}^{\kappa,\kappa}$-equivalent. By Claim 2.11.2 there exists $\gamma < \lambda^+$ such that $F(\eta)_\gamma^{-1}[1]\Delta F(\xi)_\gamma^{-1}[1]$ is stationary. Therefore, either $F(\eta)_\gamma^{-1}[1] \setminus F(\xi)_\gamma^{-1}[1]$ or $F(\xi)_\gamma^{-1}[1] \setminus F(\eta)_\gamma^{-1}[1]$ is stationary. Without loss of generality, let us assume that $F(\eta)_\gamma^{-1}[1] \setminus F(\xi)_\gamma^{-1}[1]$ is stationary. Since $S_\lambda^\kappa \circ$-reflects to $S_\gamma$, $A = \{ \alpha \in S_\gamma \mid (F(\eta)_\gamma^{-1}[1] \setminus F(\xi)_\gamma^{-1}[1]) \cap \alpha = D_\alpha^\gamma \}$ is stationary and $D_\alpha^\gamma$ is stationary in $\alpha$, and therefore $A \subseteq F(\eta)^{-1}[1]$. On the other hand, for every $\alpha$ in $A$ we have $F(\xi)^{-1}[1] \cap D_\alpha^\gamma = \emptyset$, so $A \cap F(\xi)^{-1}[1] = \emptyset$ and we conclude that $A \subseteq F(\eta)^{-1}[1] \Delta F(\xi)^{-1}[1]$. Therefore $F(\eta)^{-1}[1] \Delta F(\xi)^{-1}[1]$ is stationary, and $F(\eta)$ and $F(\xi)$ are not $E_{\lambda^+}^{2,\kappa,\kappa}$-equivalent.

Suppose $F(\eta)$ and $F(\xi)$ are not $E_{\lambda^+}^{2,\kappa,\kappa}$-equivalent, so $F(\eta)^{-1}[1] \Delta F(\xi)^{-1}[1]$ is stationary. Since $\lambda^+ < \kappa$, by Fodor’s lemma we know that there exists $\gamma < \lambda^+$ such that \{\alpha \in S_\gamma \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\} is stationary. Hence, the symmetric difference of \{\alpha \in S_\gamma \mid F(\eta)_\gamma^{-1}[1] \cap D_\alpha^\gamma \text{ stationary in } \alpha\} and \{\alpha \in S_\gamma \mid F(\xi)_\gamma^{-1}[1] \cap D_\alpha^\gamma \text{ stationary in } \alpha\} is stationary. For simplicity, let us denote by $A_\eta$ and $A_\xi$ the sets involved in this symmetric difference (i.e. $A_\eta = \{ \alpha \in S_\gamma \mid F(\eta)_\gamma^{-1}[1] \cap D_\alpha^\gamma \text{ stationary in } \alpha\}$ and $A_\xi = \{ \alpha \in S_\gamma \mid F(\xi)_\gamma^{-1}[1] \cap D_\alpha^\gamma \text{ stationary in } \alpha\}$)

Therefore, either $A_\eta \setminus A_\xi$ or $A_\xi \setminus A_\eta$ is stationary. Without loss of generality we can
assume that $A_\eta \setminus A_\xi$ is stationary. Hence, $\bigcup_{\alpha \in A_\eta \setminus A_\xi} (\mathcal{F}(\eta)_\gamma^{-1}[1] \cap D_\alpha^\gamma) \setminus \mathcal{F}(\xi)_\gamma^{-1}[1]$ is stationary and is contained in $\mathcal{F}(\eta)_\gamma^{-1}[1] \Delta \mathcal{F}(\xi)_\gamma^{-1}[1]$. By Claim 2.11.2 we conclude that $\eta$ and $\xi$ are not $E^{\kappa, \kappa}_{\lambda}$-equivalent.

Notice that Theorem 2.11 implies the consistency of

\[ E^{2}_{\lambda \text{-club}} \leq_c E^{++}_{\lambda \text{-club}} \leq_c E^{2}_{\lambda^+ \text{-club}} \leq_c E^{++}_{\lambda^+ \text{-club}}. \]

In particular, for $\lambda = \omega$ we get the expression $E^{2}_{\omega \text{-club}} \leq_c E^{\omega_2}_{\omega \text{-club}} \leq_c E^{2}_{\omega_1 \text{-club}} \leq_c E^{\omega_2}_{\omega_1 \text{-club}}$.

**Question 2.12.** Is it consistent that $E^{\omega_2}_{\gamma \text{-club}} \leq_c E^{\kappa}_{\gamma \text{-club}} \leq_c E^{2}_{\lambda \text{-club}}$ holds for all $\gamma, \lambda < \kappa$ and $\gamma < \lambda$?

We will finish this section by showing that the reduction $E^{\omega_2}_{\omega \text{-club}} \leq_c E^{\omega_2}_{\omega_1 \text{-club}}$ can be obtained using other reflection principles. Specifically, full reflection implies this reduction. For stationary subsets $S$ and $A$ of $\kappa$, we say that $S$ reflects fully in $A$ if the set $\{ \alpha \in A \mid S \cap \alpha$ is non-stationary in $\alpha \}$ is non-stationary. Notice that if $S \subset S^\kappa_\gamma$ reflects fully in $S^\kappa_\lambda$, then the set $\{ \alpha \in S^\kappa_\lambda \mid S \cap \alpha$ is stationary in $\alpha \}$ is a stationary set.

**Theorem 2.13** ([JS], Theorem 1.3). Let $\kappa_2 < \kappa_3 < \cdots < \kappa_n < \cdots$ be a sequence of supercompact cardinals. There is a generic extension $V[G]$ in which $\kappa_n = \aleph_n$ for all $n \geq 2$ and such that:

(i) Every stationary set $S \subset S^\omega_{\omega_1}$ reflects fully in $S^\omega_{\omega_1}$.

(ii) For every $2 < n$ and every $0 \leq k \leq n - 3$, every stationary set $S \subset S^\omega_n$ reflects fully in $S^\omega_{\omega_{n-1}}$.

In the generic extension of 2.13 it holds that $\omega_i^{<\omega_i} = \omega_i$ for all $i < \omega$ (see [[JS], Theorem 1.3]).

**Corollary 2.14.** The following statement is consistent: $E^{\omega_2}_{\omega \text{-club}} \leq_c E^{\omega_2}_{\omega_1 \text{-club}}$, and for every $2 < n$ and every $0 \leq k \leq n - 3$, $E^{\omega_n}_{\omega_k \text{-club}} \leq_c E^{\omega_n}_{\omega_{n-1} \text{-club}}$.

In [JS] it was also proved that Theorem 2.13 (ii) is optimal, in the sense that it cannot be improved to include the case $k = n - 2$ [JS, Proposition 1.6]. The best possible reduction we can get using only full reflection is the one in Corollary 2.14. By a $\Sigma^1_1$-completeness result, it is known that the following is consistent:

$$\forall k < n - 1 \ (E^{\omega_n}_{\omega_k \text{-club}} \leq_c E^{\omega_n}_{\omega_{n-1} \text{-club}}),$$

see Theorem 3.1 below.
3 \( \Sigma^1_1 \)-completeness

An equivalence relation \( E \) on \( X \in \{ \kappa^\kappa, 2^\kappa \} \) is \( \Sigma^1_1 \) if \( E \) is the projection of a closed set in \( X^2 \times \kappa^\kappa \) and it is \( \Sigma^1_1 \)-complete if every \( \Sigma^1_1 \) equivalence relation is Borel reducible to it. The study of \( \Sigma^1_1 \) and \( \Sigma^1_1 \)-complete equivalence relations is an important area of generalised descriptive set theory, because e.g. the isomorphism relation on classes of models is always \( \Sigma^1_1 \). The same holds, in fact, in classical descriptive set theory, but the behaviour of \( \Sigma^1_1 \) complete relations there is different. For example, in the classical setting (\( \kappa = \omega \)) the isomorphism relation is never \( \Sigma^1_1 \)-complete, while in generalised descriptive set theory this is often the case (see for example [HK, FHK]).

**Theorem 3.1** ([HK], Theorem 7). Suppose \( V = L \) and \( \kappa > \omega \). Then \( E^{\kappa, \kappa}_\mu \)-club is \( \Sigma^1_1 \)-complete for every regular \( \mu < \kappa \).

We know that \( E^{\kappa, \kappa}_{\mu\text{-club}} \upharpoonright \alpha \) is an equivalence relation for every \( \alpha < \kappa \) with \( cf(\alpha) > \lambda \). Let us define the following relation:

\[
(\eta, \xi) \in E^{\kappa, \kappa}_{\reg} \upharpoonright \alpha \iff \{ \beta \in \text{reg}(\alpha) \mid \eta(\beta) \neq \xi(\beta) \} \text{ is not stationary.}
\]

It is easy to see that \( E^{\kappa, \kappa}_{\reg} \upharpoonright \alpha \) is an equivalence relation for every Mahlo cardinal \( \alpha < \kappa \).

**Definition 3.2** (S–dual diamond). Suppose \( S \subseteq \kappa \) is a stationary set. We say that \( \kappa \) has the \( S \)-dual diamond if: There is a sequence \( \langle f_\alpha \rangle_{\alpha < \kappa} \) such that

- \( f_\alpha : \alpha \to \alpha \) for all \( \alpha \),
- if \( (Z, g) \) is a pair such that \( Z \subseteq S \) is stationary and \( g \in \kappa^\kappa \), then the set

\[
\{ \alpha \in \text{reg}(\kappa) \mid g \upharpoonright \alpha = f_\alpha \land Z \cap \alpha \text{ is stationary} \}
\]

is stationary.

It is clear that if \( S' \supseteq S \), then the \( S' \)-dual diamond implies the \( S \)-dual diamond. Notice that the \( S \)-dual diamond has a set version that is equivalent to it:

Suppose \( S \subseteq \kappa \) is a stationary set. We say that \( \kappa \) has the set version \( S \)-dual diamond if: There is a sequence \( \langle A_\alpha \rangle_{\alpha < \kappa} \) such that

- \( A_\alpha \subseteq \alpha \) for all \( \alpha \),
• if \((Z, X)\) is a pair such that \(Z \subset S\) is stationary and \(X \subseteq \kappa\), then the set
\[
\{\alpha \in \text{reg}(\kappa) \mid X \cap \alpha = A_\alpha \wedge Z \cap \alpha \text{ is stationary}\}
\]
is stationary.

It is clear, using characteristic functions, that the existence of an \(S\)-dual diamond sequence in the sense of Definition 3.2 implies this set version of \(S\)-dual diamond. For the other implication, it is easy to check that if \(\langle A_\alpha \rangle_{\alpha < \kappa}\) witnesses the set version of \(D\)-dual diamond, \(<^*\) is the canonical well order of \(\kappa \times \kappa\) and \(f : \kappa \to \kappa \times \kappa\) is the corresponding order-isomorphism, then \(B_\alpha = \{f(\beta) \mid \beta \in A_\alpha\}\) is such that: if \((Z, X)\) is a pair such that \(Z \subset S\) is stationary and \(X \subseteq \kappa \times \kappa\), then the set
\[
\{\alpha \in \text{reg}(\kappa) \mid X \cap \alpha \times \alpha = B_\alpha \wedge Z \cap \alpha \text{ is stationary}\}
\]
is stationary. Since every \(g \in \kappa^\kappa\) is a subset of \(\kappa \times \kappa\), the sequence \(\langle f_\alpha \rangle_{\alpha < \kappa}\) can be constructed from the sequence \(\langle B_\alpha \rangle_{\alpha < \kappa}\).

**Theorem 3.3.** Suppose \(S = S^\kappa_\lambda\) for some \(\lambda\) regular cardinal, or \(S = \text{reg}(\kappa)\) and \(\kappa\) is a weakly compact cardinal. If \(\kappa\) has the \(S\)-dual diamond, then \(E^{\kappa, \kappa}_S \leq E^{2, \kappa}_{\text{reg}}\).

**Proof.** Let \(\langle f_\alpha \rangle_{\alpha < \kappa}\) be a sequence that witnesses the \(S\)-dual diamond. Let \(g_\alpha : \kappa \to \kappa\) be the function defined by \(g_\alpha \upharpoonright \alpha = f_\alpha\) and \(g_\alpha(\beta) = 0\) for all \(\beta \geq \alpha\). Let us define \(F : \kappa^\kappa \to 2^\kappa\) by
\[
F(\eta)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \text{reg}(\kappa), E_S \upharpoonright \alpha \text{ is an equivalence relation, and } (\eta, g_\alpha) \in E_S \upharpoonright \alpha \\ 0 & \text{otherwise.} \end{cases}
\]

Let us prove that if \((\eta, \xi) \in E_S\), then \((F(\eta), F(\xi)) \in E^{2, \kappa}_{\text{reg}}\). Suppose \((\eta, \xi) \in E_S\). Note that \(F(\eta)(\alpha) = F(\xi)(\alpha) = 0\) for all \(\alpha \notin \text{reg}(\kappa)\), so it is sufficient to show that the set
\[
\{\alpha \in \text{reg}(\kappa) \mid F(\eta)(\alpha) \neq F(\xi)(\alpha)\}
\]
is non-stationary. Now, there is a club \(D\) such that \(D \cap \{\alpha \in S \mid \eta(\alpha) \neq \xi(\alpha)\}\) is non-stationary. So, letting \(C\) be the club of the limit points of \(D\), it holds that for all \(\alpha \in C \cap \text{reg}(\kappa)\), the functions \(\eta\) and \(\xi\) are \(E_S \upharpoonright \alpha\)-equivalent. Thus, by the definition of \(F\), at the points of the set \(C \cap \text{reg}(\kappa)\) the functions \(F(\eta)\) and \(F(\xi)\) will get the same value.

Now let us prove that if \((\eta, \xi) \notin E_S\), then \((F(\eta), F(\xi)) \notin E^{2, \kappa}_{\text{reg}}\). Suppose that \((\eta, \xi) \notin E_S\). Then there is a stationary \(Z \subset S\) on which \(\eta(\alpha) \neq \xi(\alpha)\). By the
definition of $S$–dual diamond, there is a stationary set $A \subseteq \text{reg}(\kappa)$ such that for all $\alpha \in A$ we have that $Z \cap \alpha$ is stationary and $\eta \upharpoonright \alpha = f_\alpha$. This means that

$$\{ \beta < \alpha \mid \eta(\beta) \neq \xi(\beta) \}$$

is stationary, and so $(\eta, \xi) \notin E_S \mid \alpha$ holds for all $\alpha \in A$. However $\eta \upharpoonright \alpha = f_\alpha$ implies that $(\eta, g_\alpha) \in E_S \mid \alpha$, and so by transitivity $(\xi, g_\alpha) \notin E_S \mid \alpha$. Hence we get that $F(\eta)(\alpha) = 1$, but $F(\xi)(\alpha) = 0$. This holds for all $\alpha \in A$ and $A$ is stationary, so $(F(\eta), F(\xi)) \notin E_{2, \kappa}^{\text{reg}}$. \hfill \Box

**Lemma 3.4.** Suppose $V = L$ and $\kappa$ is a weakly compact cardinal. Then $\kappa$ has the $S_\omega^{\kappa}$–dual diamond.

**Proof.** It is shown in the proof of [FHK, Theorem 55(A)] that $S_\omega^{\kappa}$ strongly reflects to $S_{\text{reg}}^{\kappa}$ (Definition 2.1). The rest of the proof is a straightforward modification of the proof of [FHK, Theorem 59], but we give it here for the sake of completeness.

We will show that there is a sequence $\langle D_\alpha, f_\alpha \rangle_{\alpha < \kappa}$ such that

- $D_\alpha \subset \alpha$ is stationary in $\alpha$ for all $\alpha$,
- $f_\alpha : \alpha \rightarrow \alpha$ for all $\alpha$,
- if $(Z, g)$ is a pair such that $Z \subset S_\omega^{\kappa}$ is stationary and $g \in \kappa^\kappa$, then the set
  $$\{ \alpha \in \text{reg}(\kappa) \mid g \upharpoonright \alpha = f_\alpha \land Z \cap \alpha = D_\alpha \}$$
  is stationary.

It is clear that this implies that $\kappa$ has the $S_\omega^{\kappa}$-dual diamond.

For the purpose of the proof we define a triple $\langle D_\alpha, f_\alpha, C_\alpha \rangle$. Suppose that $\langle D_\beta, f_\beta, C_\beta \rangle$ is already defined for $\beta < \alpha$.

Now define $\langle D, f, C \rangle$ to be the $\leq_L$-least triple such that

- $D \subset \alpha \cap S_\omega^{\kappa}$ is stationary,
- $f : \alpha \rightarrow \alpha$,
- $C$ is the intersection with $S_{\text{reg}}^{\kappa}$ of a closed and unbounded subset of $\alpha$,
- for all $\beta < \alpha$, $D \cap \beta \neq D_\beta$ or $f \upharpoonright \beta \neq f_\beta$. 

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and set $D_\alpha = D$, $f_\alpha = f$, $C_\alpha = C$ if such exists, and $D_\alpha = f_\alpha = C_\alpha = \emptyset$ otherwise.

Now our assumption is that there is a counterexample to the theorem, so let $(Z, g, C)$ be the $\leq L$-least counterexample. Let $M$ be an elementary submodel of $L_\lambda$, for some regular $\lambda > \kappa$, such that

- $|M| < \kappa$,
- $\alpha = M \cap \kappa \in C$,
- $Z \cap \alpha$ is stationary in $\alpha$, and
- $Z, g, C, S^\kappa_\omega, S^\kappa_{\text{reg}}, \kappa \in M$.

$M$ exists by the $\Pi^1_1$-reflection of the weakly compact $\kappa$. Now take the Mostowski collapse $G: M \rightarrow L_\gamma$, for some $\gamma > \alpha$. Now $G(Z) = Z \cap \alpha$, $G(g) = g | \alpha$, $G(C) = C \cap \alpha$, $G(\kappa) = \alpha$, and the sequence $\langle D_\beta, f_\beta \rangle_{\beta < \alpha}$ is definable in $L_\gamma$.

Let $\varphi(D, g, C, \kappa)$ be a formula that says "$(D, g, C)$ is the $\leq L$-least triple such that

- $D \subset S^\kappa_\omega$ is stationary,
- $g: \kappa \rightarrow \kappa$,
- $C$ is the intersection with $S^\kappa_{\text{reg}}$ of a cub of $\kappa$, and
- for all $\beta < \kappa$, $D \cap \beta \neq D_\beta$ or $f | \beta \neq f_\beta$.

But this formula relativises to $L_\gamma$ and all notions are sufficiently absolute. When relativised, it says that $(D, g)$ reflects to $\alpha \in C$, which contradicts the assumption that $(D, g, C)$ was a counterexample.

**Corollary 3.5.** Suppose $V = L$ and $\kappa$ is weakly compact. Then $E^{2,\kappa}_{\text{reg}}$ is $\Sigma^1_1$-complete.

**Proof.** This follows from Theorem 3.1, Lemma 3.4 and Theorem 3.3. \qed

**Theorem 3.6.** Suppose $\kappa$ is a supercompact cardinal. There is a generic extension $V[G]$ in which $E^{\kappa,\kappa}_{\text{reg}} \leq E^{2,\kappa}_{\text{reg}}$ holds and where $\kappa$ is still supercompact.

**Proof.** By Theorem 3.3, it is enough to find a forcing extension in which $\kappa$ has the $\text{reg}(\kappa)$–dual diamond.

In [Lav] it is proved that if $\kappa$ is a supercompact cardinal, then there is a forcing extension in which $\kappa$ remains supercompact upon forcing with any $\kappa$–directed closed forcing. Let us denote by $V[H]$ this forcing extension.
Now we will find a forcing extension of $V[H]$ in which $\kappa$ has the $\text{reg}(\kappa)$–dual diamond. In fact, we will show something stronger, we will show that there is a forcing extension in which $\kappa$ has the $\kappa$–dual diamond. Working in $V[H]$, let $\mathbb{P} = \{ f : \alpha \to \mathcal{P}(\alpha) \mid \alpha < \kappa \}$ ordered by: $p \leq q$ if $q \subseteq p$. It is easy to see that $\mathbb{P}$ is $\kappa$-directed closed, and thus $\mathbb{P} \Vdash \kappa$ is supercompact. We will prove that $\mathbb{P}$ forces that $\kappa$ has the $\kappa$–dual diamond. Suppose, towards a contradiction, that there is $G$ a $\mathbb{P}$-generic over $V[H]$ such that $\kappa$ does not have the $\kappa$–dual diamond in $V[H][G]$. Let $p \in G$, $\dot{S}, \dot{X}$ be such that $p$ forces that the sequence $\{ D_\alpha = (\bigcup G)(\alpha) \mid \alpha < \kappa \}$ does not guess $\dot{S}, \dot{X}$ as wanted, i.e.,

$$p \Vdash \dot{S} \text{ is stationary, } \dot{X} \subseteq \kappa, \text{ and the sequence } \langle D_\alpha \rangle_{\alpha < \kappa} \text{ does not guess } \dot{X} \cap \alpha \text{ in any } \alpha \text{ such that } \dot{S} \cap \alpha \text{ is stationary.}$$

We will show that the set $\{ q < p \mid q \Vdash \exists \alpha (D_\alpha = \dot{X} \cap \alpha \wedge \dot{S} \cap \alpha \text{ is stationary}) \}$ is dense below $p$, which is a contradiction. There is a club $C \subseteq \kappa$ in $V[H]$ such that for all $\alpha \in C$ it holds that $p \in \bigcup G \upharpoonright \alpha$, and $\bigcup G \upharpoonright \alpha$ decides $\dot{X} \cap \alpha$ and $\dot{S} \cap \alpha$. Since $C \in V[H]$, we have that $C \in V[H][G]$. On the other hand, $\kappa$ is $\Pi^1_1$-indecomposable in $V[H][G]$, and the sentence:

$$(C \text{ is unbounded in } \kappa) \land (\dot{S}_C \text{ is stationary in } \kappa) \land (\kappa \text{ is regular})$$

is a $\Pi^1_1$ property of the structure $(V_{\kappa}^{V[H][G]}, \in, \dot{S}_C, C)$. By $\Pi^1_1$-reflection, there is $\alpha < \kappa$ in $V[H][G]$ such that $C \cap \alpha$ is unbounded, and hence $\alpha \in C$, $\dot{S}_C \cap \alpha$ is stationary in $\alpha$, and $\alpha$ is regular. Since $\mathbb{P}$ is $<\kappa$-closed, we have that $\dot{X} \cap \alpha \in V[H]$. Let $q$ be the condition $\bigcup G \upharpoonright \alpha \cup \{ (\alpha, \dot{X} \cap \alpha) \}$. Clearly $q < p$ and $q \Vdash \exists \alpha (D_\alpha = \dot{X} \cap \alpha \wedge \dot{S} \cap \alpha \text{ is stationary})$ as we wanted. 

\begin{theorem}
If $\kappa$ is a $\Pi^1_2$-indecomposable cardinal, then $E_{reg}^{\kappa} = \Sigma^1_1$-complete.
\end{theorem}

\textbf{Remark.} Here the notion of $\Pi^1_2$–indecomposability is the usual one, not to be confused with the $\Pi^1_1$–indecomposability from Definition 2.2.

\textbf{Proof.} Let $E$ be a $\Sigma^1_1$ equivalence relation on $\kappa^\kappa$. Then there is a closed set $C$ on $\kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ such that $\eta \in E \xi$ if and only if there exists $\theta \in \kappa^\kappa$ such that $(\eta, \xi, \theta) \in C$. Let us define $U = \{ (\eta \upharpoonright \alpha, \xi \upharpoonright \alpha, \theta \upharpoonright \alpha) \mid (\eta, \xi, \theta) \in C \cap \alpha < \kappa \}$, and for every $\gamma < \kappa$ define $C_\gamma = \{ (\eta, \xi, \theta) \in \gamma^\gamma \times \gamma^\gamma \times \gamma^\gamma \mid \forall \alpha < \gamma (\eta \upharpoonright \alpha, \xi \upharpoonright \alpha, \theta \upharpoonright \alpha) \in U \}$. Let $E_\gamma \subseteq \gamma^\gamma \times \gamma^\gamma$ be the relation defined by $(\eta, \xi) \in E_\gamma$ if and only if there exists $\theta \in \gamma^\gamma$ such that $(\eta, \xi, \theta) \in C_\gamma$. Since $E$ is an equivalence relation, it follows that $E_\gamma$ is reflexive and symmetric, but not necessary transitive. Let us define the reduction by

$$F(\eta)(\alpha) = \begin{cases} f_\alpha(\eta) & \text{if } E_\alpha \text{ is an equivalence relation and } \eta \upharpoonright \alpha \in \alpha^\alpha \\ 0 & \text{otherwise.} \end{cases}$$
where \( f_\alpha(\eta) \) is a code in \( \kappa \setminus \{0\} \) for the \( E_\kappa \)-equivalence class of \( \eta \).

Let us prove that if \((\eta, \xi) \in E\), then \((F(\eta), F(\xi)) \in E^\kappa_{reg}\). Suppose \((\eta, \xi) \in E\). Then there is \( \theta \in \kappa \) such that \((\eta, \xi, \theta) \in C\) and for all \( \alpha < \kappa \) we have that 
\( (\eta \upharpoonright \alpha, \xi \upharpoonright \alpha, \theta \upharpoonright \alpha) \in U \). On the other hand, we know that there is a club \( D \) such that for all \( \alpha \in D \cap reg(\kappa) \), \( \eta \upharpoonright \alpha \upharpoonright \alpha \in \alpha^\alpha \). We conclude that for all \( \alpha \in D \cap reg(\kappa) \), if \( E_\alpha \) is an equivalence relation, then \((\eta, \xi) \in E_\alpha\). Therefore, for all \( \alpha \in D \cap reg(\kappa) \), 
\( F(\eta)(\alpha) = F(\xi)(\alpha) \), so \((F(\eta), F(\xi)) \in E^\kappa_{reg}\). Let us prove that if \((\eta, \xi) \notin E\), then \((F(\eta), F(\xi)) \notin E^\kappa_{reg}\). Suppose \( \eta, \xi \in \kappa \) are such that \((\eta, \xi) \notin E\). We know that there is a club \( D \) such that for all \( \alpha \in D \cap reg(\kappa) \), \( \eta \upharpoonright \alpha \upharpoonright \alpha \in \alpha^\alpha \).

Notice that because \( C \) is closed \((\eta, \xi) \notin E\) is equivalent to
\[ \forall \theta \in \kappa^\kappa (\exists \alpha < \kappa (\eta \upharpoonright \alpha, \xi \upharpoonright \alpha, \theta \upharpoonright \alpha) \notin U) , \]
so the sentence \((\eta, \xi) \notin E\) is a \( \Pi^1_1 \) property of the structure \((V_\kappa, \in, U, \eta, \xi)\). On the other hand, the sentence \( \forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[(\zeta_1, \zeta_2) \in E \land (\zeta_2, \zeta_3) \in E] \rightarrow (\zeta_1, \zeta_3) \in E\) is equivalent to the sentence \( \forall \zeta_1, \zeta_2, \zeta_3, \theta_1, \theta_2 \in \kappa^\kappa[\exists \zeta_3 \in \kappa^\kappa(\psi_1 \lor \psi_2 \lor \psi_3)]\), where \( \psi_1, \psi_2 \) and \( \psi_3 \) are, respectively, the formulas \( \exists \alpha_1 < \kappa (\zeta_1 \upharpoonright \alpha_1, \zeta_2 \upharpoonright \alpha_1, \theta_1 \upharpoonright \alpha_1) \notin U\), \( \exists \alpha_2 < \kappa (\zeta_2 \upharpoonright \alpha_2, \zeta_3 \upharpoonright \alpha_2, \theta_2 \upharpoonright \alpha_2) \notin U\), and \( \forall \alpha_3 < \kappa (\zeta_1 \upharpoonright \alpha_3, \zeta_3 \upharpoonright \alpha_3, \theta_3 \upharpoonright \alpha_3) \in U\). Therefore, the sentence \( \forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[(\zeta_1, \zeta_2) \in E \land (\zeta_2, \zeta_3) \in E] \rightarrow (\zeta_1, \zeta_3) \in E\) is a \( \Pi^1_2 \) property of the structure \((V_\kappa, \in, U)\). It follows that the sentence
\[ (D \text{ is unbounded in } \kappa) \land ((\eta, \xi) \notin E) \land (E \text{ is an equivalence relation}) \land (\kappa \text{ is regular}) \]
is a \( \Pi^1_2 \) property of the structure \((V_\kappa, \in, U, \eta, \xi)\). By \( \Pi^1_2 \) reflection, we know that there are stationary many \( \gamma \in reg(\kappa) \) such that \( \gamma \) is a limit point of \( D \), \( E_\gamma \) is an equivalence relation, and \((\eta \upharpoonright \gamma, \xi \upharpoonright \gamma) \notin E_\gamma\). We conclude that there are stationary many \( \gamma \in reg(\kappa) \) such that \( f_\gamma(\eta) \neq f_\gamma(\xi) \), and hence \((F(\eta), F(\eta)) \notin E^\kappa_{reg}\). \( \blacksquare \)

**Corollary 3.8.** Suppose \( \kappa \) is a supercompact cardinal. There is a generic extension \( V[G] \) in which \( E^\kappa_{reg} \) is \( \Sigma^1_1 \)-complete.

**Proof.** Let \( V[G] \) be the generic extension of Theorem 3.6. Since \( \kappa \) is supercompact in \( V[G] \), it is \( \Pi^1_2 \)-indescribable. By Theorem 3.7, \( E^\kappa_{reg} \) is \( \Sigma^1_1 \)-complete, and by Theorem 3.6 we know that \( E^\kappa_{reg} \preceq E^2_{reg} \). We conclude that \( E^2_{reg} \) is \( \Sigma^1_1 \)-complete in \( V[G] \). \( \blacksquare \)

Let \( NS \) denote the equivalence on \( 2^\kappa \) modulo the non-stationary ideal, i.e. \( \eta \) \( NS \) \( \xi \) if and only if \( \eta^{-1}[1] \Delta \xi^{-1}[1] \) is not stationary. For every stationary \( S \subseteq \kappa \) the relation \( E^2_{S,\kappa} \) is continuously reducible to \( NS \). The reduction \( \mathcal{F} : 2^\kappa \rightarrow 2^\kappa \) is defined as follows:
\[ \mathcal{F}(\eta)(\alpha) = \begin{cases} \eta(\alpha) & \text{if } \alpha \in S \\ 1 & \text{otherwise} \end{cases} \]
We conclude that the statement $NS$ is $\Sigma^1_1$-complete is consistent, this follows from Corollary 3.5 (it also follows from Corollary 3.8).

We will finish this article with a result related to model theory.

**Theorem 3.9.** Let $DLO$ be the theory of dense linear orderings without end points. If $\kappa$ is a $\Pi^1_2$-indescribable cardinal, then $\cong_{DLO}$ is $\Sigma^1_1$-complete.

**Proof.** By Theorem 3.7 it is enough to show that $E^\kappa_{\text{reg}} \leq c \cong_{DLO}$. To show this, first we will construct models of $DLO$, $A^f(\kappa)$, for every $f : \kappa \to \kappa$, such that $f E^\kappa_{\text{reg}} g$ if and only if $A^{f(\kappa)} \cong A^{g(\kappa)}$. After that we construct the reduction of $E^\kappa_{\text{reg}}$ to $\cong_{DLO}$.

Let us take the language $L' = \{L, C, <, R\}$, with $L$ and $C$ as unary predicates, and $<$ and $R$ as binary relations. Let $K$ be the class of $L'$-structures $A = (\text{dom}(A), L, C, <, R)$ that satisfy the following conditions:

- $L \cap C = \emptyset$.
- $L \cup C = \text{dom}(A)$.
- $< \subseteq L \times L$ is a dense linear order without end points on $L$.
- $R \subseteq L \times C$.
- Let us denote by $R^-(y, x)$ the formula $\neg R(y, x)$. For all $x \in C$, it holds that $R(A, x) \cup R^-(A, x) = L$, $R(A, x)$ has no largest element, and $R^-(A, x)$ has no least element and they are non-empty.

Let us define the following partial order $\preceq$ on $K$. We say that $A \preceq B$ iff:

- $A \subseteq B$,
- for all $x \in C^A$, $R(B, x) = \{y \in L^B \mid \exists z \in R(A, x), y < z\}$ and $R^-(B, x) = \{y \in L^B \mid \exists z \in R^-(A, x), z < y\}$,
- for all $x \in C^B \setminus C^A$ there are $y \in R(B, x)$ and $z \in R^-(B, x)$ such that for all $a \in L^A$, $a < y \lor a > z$.

Notice that it is possible to have a chain $A_0 \preceq A_1 \preceq \cdots$ of length $\alpha$ in $K$, and a structure $C \in K$, such that $\bigcup_{i<\alpha} A_i \in K$, $A_i \preceq C$ holds for all $i < \alpha$, and $\bigcup_{i<\alpha} A_i \not\preceq C$. But all other requirements of AEC’s are satisfied, as one can easily see, in particular for every chain $A_0 \preceq A_1 \preceq \cdots$ of length $\alpha$ in $K$, $\bigcup_{i<\alpha} A_i \in K$. 

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Claim 3.9.1. \((K, \preceq)\) has the amalgamation property and the joint embedding property.

Proof. The joint embedding property is easily seen to follow from the amalgamation property. For the amalgamation property, let \(A, B, C \in K\) be such that \(A \preceq B\) and \(A \preceq C\) hold. Without loss of generality, we can assume that \(\text{dom}(B) \cap \text{dom}(C) = \text{dom}(A)\). Let us construct \(D\) with \(\text{dom}(B) \cup \text{dom}(C) = \text{dom}(D)\), \(L^D = L^B \cup L^C\), and \(C^D = C^B \cup C^C\). To define \(<^D\) and \(R^D\), first define \(<'^D\) and \(<^C\). For every two elements \(b, c \in L^D\) define \(b <^D c\) if either \(b <^C c\), or there is \(a \in L^A\) such that \(b <^A a <^C c\), or \(b \in L^B\), \(c \in L^C\) and there is no \(a \in L^A\) such that \(c <^A a <^B b\). For every \(x \in C^A\), \(R(D, x) = R(B, x) \cup R(C, x)\). For all \(x \in C^B \setminus C^A\), \(y \in R(D, x)\) if and only if there exists \(z \in L^B\) such that \(z \in R(B, x)\) and \(y <^D z\). For all \(x \in C^C \setminus C^A\), \(y \in R(D, x)\) if and only if there exists \(z \in L^C\) such that \(z \in R(C, x)\) and \(y <^D z\). It is clear that \(D \in K\), and \(B \preceq D\) and \(C \preceq D\). \(\square\)

Let us denote by \(A_1 \oplus_{A_0} A_2\) the structure \(D\), in Claim 3.9.1, that witnesses the amalgamation property for the structures \(A_0 \preceq A_1\) and \(A_0 \preceq A_2\). For every ordinal \(\alpha\), let us denote by \(\alpha^*\) the set \(\alpha\) ordered by the reverse order \(<^\ast\), i.e., \(\beta <^\ast \gamma\) if \(\gamma \in \beta\). Let us order the members of \(\mathbb{Q} \times \alpha^*\) by: \((r_1, \alpha_1) <^{\ast\alpha} (r_2, \alpha_2)\) iff \(\alpha_1 <^\ast \alpha_2\), or \(\alpha_1 = \alpha_2\) and \(r_1 <^\mathbb{Q} r_2\).

Let \(K_{<\kappa}\) be the collection of all members of \(K\) of size less than \(\kappa\). For every \(A \in K_{<\kappa}\), denote by \(\{A(i)\}_{i<\kappa}\) an enumeration of all the strong extensions of \(A\), i.e., \(A \preceq B\), of size less than \(\kappa\) (up to isomorphism over \(A\)). Let \(\Pi : \kappa \to \kappa \times \kappa\), \(\Pi(\alpha) = (pr_1(\Pi(\alpha)), pr_2(\Pi(\alpha)))\) be a bijection such that \(pr_1(\Pi(i)) \leq i\) for all \(i\). Given a function \(f : \kappa \to \text{reg}(\kappa)\), let us construct the following sequence of models:

- \(A^f_0 = (\mathbb{Q}, \varnothing, <, \varnothing)\).

- For a successor ordinal, let \(D = A^f_i \oplus_{A^f_{pr_1(\Pi(i))}} A^f_{pr_1(\Pi(i))}(pr_2(\Pi(i)))\). Define \(L^{A^f_{i+1}} = L^D \cup \mathbb{Q}\), \(C^{A^f_{i+1}} = C^D\), \(<^{A^f_{i+1}} = <^D \cup <^\mathbb{Q} \cup \{(x, y) \mid x \in L^D \land y \in \mathbb{Q}\}\), and \(R^{A^f_{i+1}} = R^D\). Clearly \(A^f_{i+1} \in K\).

- For \(i\) a limit ordinal, let \(D = \bigcup_{j<i} A^f_j\). Define \(L^{A^f_i} = L^D \cup (\mathbb{Q} \times f(i)^*)\), \(C^{A^f_i} = C^D \cup \{x\}, <^{A^f_i} = <^D \cup <^\mathbb{Q} \cup \{(a, b) \mid a \in L^D \land b \in \mathbb{Q} \times f(i)^*\}\), and \(R^{A^f_i} = R^D \cup \{(y, x) \mid y \in L^D\}\). Clearly \(A^f_i \in K\).

Define \(A^f_\kappa\) by \(\bigcup_{j<\kappa} A^f_j\). Then \(A^f_\kappa = (L^{A^f_\kappa}, <^{A^f_\kappa})\) is a model of DLO.
Notice that if \( i < \kappa \) and \( C \in K, |C| < \kappa \), are such that \( A^f_i \preceq C \), then there is \( j < \kappa \) such that \( A^f_j(j) = C \). Therefore there is \( l < \kappa \) such that \( \Pi(l) = (i, j) \), \( A^f_{\text{pr}_1(\Pi(l))} = A^f_i \), and \( A^f_{\text{pr}_1(\Pi(l))(\text{pr}_2(\Pi(l)))} = C \). We conclude that if \( i < \kappa \) and \( C \in K_{<\kappa} \) are such that \( A^f_i \preceq C \), then there is \( j < \kappa \) and a strong embedding \( F : C \to A^f_j \) such that \( F(C) \preceq A^f_j \) and \( F \upharpoonright A^f_i = id \). Now we will show that if \( f \) and \( g \) are functions from \( \kappa \) into \( \text{reg}(\kappa) \) such that \( f \upharpoonright (\kappa \setminus \text{reg}(\kappa)) = g \upharpoonright (\kappa \setminus \text{reg}(\kappa)) \), then \( f \sim_{\text{reg}} g \) if and only if \( A^f \cong A^g \). First of all, let us prove that \( (f, g) \in E^\kappa_{\text{reg}} \) implies \( A^f \cong A^g \). Suppose \( (f, g) \in E^\kappa_{\text{reg}} \). Then there is a club \( C \) such that for all \( \alpha \in C \setminus \text{reg}(\kappa) \), \( f(\alpha) = g(\alpha) \). Since \( f \upharpoonright (\kappa \setminus \text{reg}(\kappa)) = g \upharpoonright (\kappa \setminus \text{reg}(\kappa)) \), we have that for all \( \alpha \in C \), \( f(\alpha) = g(\alpha) \). By the way the models \( A^f_\alpha \) and \( A^g_\alpha \) were constructed for \( \alpha \) a limit ordinal, we know that if \( \alpha \) is such that \( f(\alpha) = g(\alpha) \) and there is an isomorphism \( F : \bigcup_{i < \alpha} A^f_i \to \bigcup_{i < \alpha} A^g_i \), then there is an isomorphism \( G : A^f_\alpha \to A^g_\alpha \) such that \( F \subseteq G \). For all \( i < \kappa \) construct \( \alpha_i < \kappa \) and a strong embedding \( F_i \) such that the following hold:

(i) For every \( i < \kappa \) there is some \( \gamma \in C \) such that \( \alpha_i < \gamma < \alpha_{i+1} \).

(ii) For all \( i < j < \kappa \), \( f_i \subseteq f_j \).

(iii) The following holds for every limit ordinal \( \beta < \kappa \):

- for every even \( 0 < i < \omega \), \( \text{dom}(F_{\beta+i}) = A^f_{\alpha_{\beta+i-1}} \) and \( F_{\beta+i}(A^f_{\alpha_{\beta+i-1}}) \preceq A^g_{\alpha_{\beta+i}}, \)
- for every odd \( 0 < i < \omega \), \( \text{rang}(F_{\beta+i}) = A^g_{\alpha_{\beta+i}}, \) and \( F_{\beta+i}^{-1}(A^g_{\alpha_{\beta+i}}) \preceq A^f_{\alpha_{\beta+i}}, \)
- for \( i = 0, \alpha_\beta = \bigcup_{1 < \beta} \alpha_i, \text{dom}(F_\beta) = A^f_{\alpha_\beta}, \) and \( \text{rang}(F_\beta) = A^g_{\alpha_\beta}. \)

We will construct these sequences by induction. For \( i = 0 \), take \( \alpha_0 = 0 \) and \( F_0 = id \).

Successor case: Suppose \( \beta \) is a limit ordinal or zero, and \( 0 \leq i < \omega \) are such that \( \alpha_{\beta+i} \) and \( F_{\beta+i} \) are constructed such that (i), (ii), and (iii) are satisfied. Let us start with the case when \( i \) is odd. Choose \( \alpha_{\beta+i+1} \) such that (i) holds. Since \( F_{\beta+i}^{-1}(A^g_{\alpha_{\beta+i}}) \preceq A^f_{\alpha_{\beta+i+1}} \), there are \( C \in K_{<\kappa} \) and \( F \supseteq F_{\beta+i} \) such that \( A^g_{\alpha_{\beta+i}} \preceq C \) and \( F : A^f_{\alpha_{\beta+i+1}} \to C \) is an isomorphism. By the observation we made above, there is \( j < \kappa \) and a strong embedding \( G : C \to A^f_j \) such that \( G(C) \preceq A^f_j \) and \( G \upharpoonright A^g_{\alpha_{\beta+i}} = id \). Define \( F_{\alpha_{\beta+i+1}} = G \circ F_{\alpha_{\beta+i}} \). Clearly \( F_{\alpha_{\beta+i+1}} \) satisfies conditions (ii) and (iii). The case when \( i \) is even is similar to the odd case.

Limit case: Suppose \( \beta \) is a limit ordinal such that for all \( i < \beta \), \( \alpha_i \) and \( F_i \) are constructed such that (i), (ii), and (iii) are satisfied. By (i), we know that
\( \alpha_\beta = \bigcup_{i < \beta} \alpha_i \) is a limit point of \( C \), so \( f(\alpha_\beta) = g(\alpha_\beta) \). On the other hand, by conditions (ii) and (iii) we know that

\[
\bigcup_{i < \beta} F_i : \bigcup_{i < \beta} \mathcal{A}_{\alpha_i}^f \to \bigcup_{i < \beta} \mathcal{A}_{\alpha_i}^g
\]
is an isomorphism. Therefore, there is an isomorphism \( G : \mathcal{A}_\alpha^f \to \mathcal{A}_\alpha^g \) such that \( \bigcup_{i < \beta} F_i \subseteq G \). We conclude that \( F_{\alpha_\beta} = G \) satisfies (ii) and (iii).

Finally, notice that

\[
\bigcup_{i < \kappa} F_i : \bigcup_{i < \kappa} \mathcal{A}_{\alpha_i}^f \to \bigcup_{i < \kappa} \mathcal{A}_{\alpha_i}^g
\]
is an isomorphism. We conclude that \( \mathcal{A}_\eta^f \) and \( \mathcal{A}_\eta^g \) are isomorphic.

Let us prove that \( \mathcal{A}_\eta^f \cong \mathcal{A}_\eta^g \). Suppose, towards a contradiction, that \( (f, g) \notin E_{reg}^{\kappa, \kappa} \) and there is an isomorphism \( F : \mathcal{A}_\eta^f \to \mathcal{A}_\eta^g \). Since \( F \) is an isomorphism, there is a club \( C \) such that \( F(\bigcup_{\alpha \in C} \mathcal{A}_\alpha^f) = \bigcup_{\alpha \in C} \mathcal{A}_\alpha^g \) holds for all \( \alpha \in C \). Since \( (f, g) \notin E_{reg}^{\kappa, \kappa} \), \( C \cap \{ \alpha \in \text{reg}(\kappa) \mid f(\alpha) \neq g(\alpha) \} \) is nonempty. Take \( \alpha \in C \cap \{ \gamma \in \text{reg}(\kappa) \mid f(\gamma) \neq g(\gamma) \} \). We know that \( F(\bigcup_{\alpha \in C} \mathcal{A}_\alpha^f) = \bigcup_{\alpha \in C} \mathcal{A}_\alpha^g \) and \( f(\alpha) \neq g(\alpha) \). Hence, the co-initiality of \( \{ a \in \mathcal{A}_\eta^f \mid \forall b \in \bigcup_{i < \alpha} \mathcal{A}_i^f (b <_{\mathcal{A}_\eta^f} a) \} \) with respect to \( <_{\mathcal{A}_\eta^f} \) is \( f(\alpha) \). Since \( F \) is an isomorphism and \( F(\bigcup_{\alpha \in C} \mathcal{A}_\alpha^f) = \bigcup_{\alpha \in C} \mathcal{A}_\alpha^g \), the co-initiality of \( \{ a \in \mathcal{A}_\eta^g \mid \forall b \in \bigcup_{i < \alpha} \mathcal{A}_i^g (b <_{\mathcal{A}_\eta^g} a) \} \) with respect to \( <_{\mathcal{A}_\eta^g} \) is also \( f(\alpha) \). We conclude that \( f(\alpha) = cf(g(\alpha)) \), so \( f(\alpha) = g(\alpha) \), a contradiction. To finish with the construction of the models, let us define \( \mathcal{A}_{\eta, \eta}^{f, \eta} \) for all \( f : \kappa \to \kappa \). Fix a bijection \( G : \kappa \to \text{reg}(\kappa) \). Define \( \mathcal{F} : \kappa^\kappa \to \kappa^\kappa \) by

\[
\mathcal{F}(f)(\alpha) = \begin{cases} 
G(f(\alpha)) & \text{if } \alpha \in \text{reg}(\kappa) \\
1 & \text{otherwise}
\end{cases}
\]

Clearly \( f \in E_{reg}^{\kappa, \kappa} \) if and only if \( \mathcal{F}(f) \in E_{reg}^{\kappa, \kappa}(\mathcal{F}(f)) \), and \( \mathcal{F}(f) \in E_{reg}^{\kappa, \kappa} \mathcal{F}(g) \) if and only if \( \mathcal{A}_{\eta, \eta}^{f, \eta} \) and \( \mathcal{A}_{\eta, \eta}^{g, \eta} \) are isomorphic. Now we will construct a reduction of \( E_{reg}^{\kappa, \kappa} \) to \( \cong_{\text{DLO}} \) by coding the models \( \mathcal{A}_{\eta, \eta}^{f, \eta} \) functions \( \eta : \kappa \to \kappa \).

Clearly the models \( \mathcal{A}_{\eta, \eta}^{f, \eta} \) satisfy that

\[
\mathcal{F}(f) \upharpoonright \alpha = \mathcal{F}(g) \upharpoonright \alpha \iff \mathcal{A}_{\eta, \eta}^{f, \eta} = \mathcal{A}_{\eta, \eta}^{g, \eta}.
\]

For every \( f \in \kappa^\kappa \) define \( C_f \subseteq \text{Card} \cap \kappa \) such that for all \( \alpha \in C_f \), it holds that for every \( \beta < \alpha \), \( |\mathcal{A}_{\beta, \eta}^{f, \eta}| < |\mathcal{A}_{\eta, \eta}^{f, \eta}| \). For every \( f \in \kappa^\kappa \) and \( \alpha \in C_f \) choose a bijection \( E_{\alpha}^{\beta} : \text{dom}(\mathcal{A}_{\eta, \eta}^{f, \eta}) \to |\mathcal{A}_{\eta, \eta}^{f, \eta}| \) such that for all \( \beta < \alpha \) in \( C_f \) it holds that \( E_{\beta}^{\beta} \subseteq E_{\alpha}^{\alpha} \).
Then \( \bigcup_{\alpha \in C} E^\alpha_f = E_f \) is such that \( E_f : \text{dom}(A^F(f)) \to \kappa \) is a bijection, and for every \( f, g \in \kappa^\kappa \) and \( \alpha < \kappa \) the following holds: If \( F(f) \upharpoonright \alpha = F(g) \upharpoonright \alpha \), then \( E_f \upharpoonright \text{dom}(A^F(f)) = E_g \upharpoonright \text{dom}(A^F(g)) \). Let \( \pi \) be the bijection in Definition 1.6. Define the function \( G \) by:

\[
G(F(f))(\alpha) = \begin{cases} 
1 & \text{if } \alpha = \pi(m, a_1, \ldots, a_n) \text{ and } A^F(f) \models P_m(E^{-1}_f(a_1), \ldots, E^{-1}_f(a_n)) \\
0 & \text{in the other case.}
\end{cases}
\]

To show that \( G \) is continuous, let \([\eta \upharpoonright \alpha]\) be a basic open set and \( \xi \in G^{-1}[\eta \upharpoonright \alpha] \). There is \( \beta \in C_\xi \) such that for all \( \gamma < \alpha \), if \( \gamma = \pi(m, a_1, a_2, \ldots, a_n) \), then \( E^{-1}_\xi(\alpha_i) \in \text{dom}(A^\xi_\beta) \) holds for all \( i \leq n \). Since for all \( \zeta \in [\xi \upharpoonright \beta] \) it holds that \( A^\xi_\zeta = A^\xi_\beta \), for every \( \gamma < \alpha \) such that \( \gamma = \pi(m, a_1, a_2, \ldots, a_n) \), it holds that

\[
A^\xi \models P_m(E^{-1}_\xi(a_1), E^{-1}_\xi(a_2), \ldots, E^{-1}_\xi(a_n))
\]

if and only if

\[
A^\zeta \models P_m(E^{-1}_\zeta(a_1), E^{-1}_\zeta(a_2), \ldots, E^{-1}_\zeta(a_n))
\]

We conclude that \( G(\xi) \in [\eta \upharpoonright \alpha] \), and \( G \circ F \) is a continuous reduction of \( E_{\kappa, \kappa}^{\text{reg}} \) to \( \approx_{\text{DLO}} \). \( \square \)

References


