Forcing locally definable well–orders of the universe without the GCH

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The general problem

For a given cardinal κ (for many cardinals κ), force the existence of a well–order $\mathcal{W}$ of $H(\kappa^+)$ such that $\mathcal{W}$ is definable over $\langle H(\kappa^+), \in \rangle$ by a formula, possibly without parameters, or with parameters of some specific kind. Do this while preserving some properties, or together with some properties holding in the model.

Some immediate observations

If $V = L[A]$ for a set A, then there is a well–order $\mathcal{W}$ of V such that $\mathcal{W} \restriction H(\kappa^+) \times H(\kappa^+)$ is a well–order of $H(\kappa^+)$ Δ_1–definable from A for every κ above the rank of A.

If κ is regular, then after adding a Cohen subset of κ there is no well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in\rangle$ from small parameters (i.e., parameters in $H(\kappa)$). And if $\kappa^{<\kappa} = \kappa$ and we add κ^+–many Cohen subsets of κ, then there is no well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in\rangle$ from any parameters.

A general positive result

Proposition (Shelah): Suppose λ is a strong limit of uncountable cofinality. Suppose for cofinally many $\eta < \lambda$ there is a well–order of $H(\eta^+)$ definable over $\langle H(\eta^+), \in\rangle$ without parameters. Then there is a well–order of $H(\lambda^+)$ definable over $\langle H(\lambda^+), \in\rangle$ without parameters.

Proof: This follows from a theorem of Shelah ("Pcf without choice," [Sh835]):

Theorem (Shelah): Suppose λ is a singular strong limit and $\mathrm{cf}(\lambda) = \kappa \geq \omega_1$. Then there is a family $\mathcal{F}_p$ ($p \in \mathcal{P}(\mathcal{P}(\kappa))$) such that

(i) for every p there is some $S \subseteq \kappa$ such that $\mathcal{F}_p$ is a collection of functions $g : S \longrightarrow \lambda$,

(ii) for every p there is a well–order of $\mathcal{F}_p$ definable in $H(\lambda^+)$ from p (and all these well–orders have the same definition), and

(iii) for every function $f : \kappa \longrightarrow \lambda$ there is a decomposition $\kappa = \bigcup_{n<\omega} S_n$ such that for all n, $f \restriction S_n \in \mathcal{F}_{p_n}$ for some $p_n \in \mathcal{P}(\mathcal{P}(\kappa))$.

Proof of the Proposition: Fix $(\lambda_\xi)_{\xi<\kappa}$ converging to λ. Given $X \subseteq \lambda$, let $f_X : \kappa \longrightarrow \lambda$ be such that $f(\xi)$ codes a pair $\langle \eta, \gamma \rangle$, where $\eta \geq \lambda_\xi$, and $X \cap \lambda_\xi$ is the γ-th member of the well-order of $H(\eta^+)$ definable in $H(\eta^+)$.

Note that X can be recovered from $f_X \upharpoonright S$ for any unbounded $S \subseteq \kappa$. Now look at the minimal $p_n \in \mathcal{P}(\mathcal{P}(\kappa))$ (in some definable well–order of $H(\eta_0^+)$, where $\mathcal{P}(\mathcal{P}(\kappa)) \in H(\eta_0^+)$) such that $f_X \upharpoonright S \in \mathcal{F}_{p_n}$ for some unbounded $S \subseteq \kappa$ and look at the index of $f_X \upharpoonright S$ in the well–order of $\mathcal{F}_{p_n}$.
$\square$

Of course the above theorem of Shelah also proves:

Proposition (Shelah): (ZFC) If λ is a singular strong limit of uncountable cofinality, then $L(V_{\lambda+1}) \models \mathrm{ZFC}$.

This is not necessarily true when $\mathrm{cf}(\lambda) = \omega$:

For example, if there is an elementary embedding $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$, then $L(V_{\lambda+1}) \models \neg \mathrm{AC}$. Otherwise a contradiction can be derived as in Kunen's proof that there is no nontrivial $j : V \longrightarrow V$ in ZFC.

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The above situation can be obtained from weaker large cardinals: Suppose κ is measurable and its measurability indestructible under $<\kappa$–directed closed forcing (this can be obtained from a supercompact cardinal by Laver). Add many Cohen subsets of κ and then make κ of cofinality ω with Prikry forcing. In the final extension $L(V_{\kappa+1}) \models \neg\mathrm{AC}$.

This situation needs some large cardinals:

Observation: (ZFC) Suppose λ is a strong limit cardinal of countable cofinality such that $L(V_{\lambda+1}) \models \neg\mathrm{AC}$. Then, $X^\sharp$ exists for every set of ordinals X bounded in λ. In fact, for every such X there is an inner model containing X and a measurable cardinal.

[Proof straightforward using Jensen's covering lemma and Dodd-Jensen's covering lemma, resp.]

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[Proof straightforward using Jensen's covering lemma and Dodd-Jensen's covering lemma, resp.]

Question: What is the consistency strength of ZFC + There is a strong limit cardinal λ of countable cofinality such that $L(V_{\lambda+1}) \models \neg\text{AC}$?

$H(\omega_1)$

In the presence of large cardinals, one cannot force, by set–forcing, the existence of a well–order of $H(\omega_1)$ definable over $\langle H(\omega_1), \in\rangle$ from parame-
ters. For example if $L(\mathbb{R}) \models \neg\mathrm{AC}$ in every set–generic extension.

So, from now on $\kappa \geq \omega_2$.

Negative results for $H(\omega_2)$

Proposition 1: Suppose A is a stationary and co-stationary $A \subseteq \omega_1$ (codes an Aronszajn tree, an ω_1–sequence of distinct reals, etc.). Suppose $L(\mathbb{R})$ satisfies the Axiom of Determinacy and there is a Woodin cardinal below a measurable cardinal. Then there is no pair $(\Phi_0(x), \Phi_1(x))$ of "necessarily incompatible" Π_2 formulas over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ such that

(i) A is defined over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ by $\Phi_0(x)$, and

(ii) $\omega_1 \setminus A$ is defined over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ by $\Phi_1(x)$.

Proposition 2: Suppose $L(\mathbb{R})$ satisfies the Axiom of Determinacy and there is a Woodin cardinal below a measurable cardinal. Then there is no "necessarily antisymmetric" Π_2 formula $\Phi(x, y)$ over $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle_{r\in\mathbb{R}}$ such that $\Phi(x, y)$ defines a well–order $\mathcal{W}$ of $\mathbb{R}$.

The proof uses the fact that $\langle H(\omega_2), \in, NS_{\omega_1}, r \rangle_{r\in\mathbb{R}}^{L(\mathbb{R})^{\mathbb{P}_{max}}}$ realises all Π_2 statements holding in V. Since $\mathbb{P}_{max} \in L(\mathbb{R})$ is homogeneous, (Φ_0, Φ_1) as in Prop. 1 would make A definable in $L(\mathbb{R})$, and $\Phi(x, y)$ as in Prop. 2 would make $\mathcal{W}$ definable in $L(\mathbb{R})$.

Small digression: Relative definability

Some combinatorial objects code always a lot of information, some not necessarily. Examples:

If $\vec{C} = (C_\delta \mid \delta \in \mathrm{Lim}(\omega_1))$ is a ladder system (i.e., for all δ, $C_\delta \subseteq \delta$ is cofinal in δ and of order type ω), then from $\vec{C}$ one can always define, over $\langle H(\omega_2), \in\rangle$,

- A special Aronszajn tree on ω_1 together with a witness that it is special.
- A Countryman line together with a witness that it is Countryman.
- A Hausdorff gap.
- A simplified $(\omega, 1)$–morass.
- $\ldots$

That is: there is a formula $\varphi(x, y)$ such that the set of a such that $\langle H(\omega_2), \in\rangle \models \varphi(a, \vec{C})$ is a special Aronszajn tree on ω_1, etc.

Also, from any of the objects in the previous list one can always define over $\langle H(\omega_2), \in\rangle$ a ladder system, a special Aronszajn tree on ω_1 together with a witness that is is special, a Countryman line together with a witness that it is Countryman, a Hausdorff gap, a simplified $(\omega, 1)$–morass, etc.

Example: Suppose C is a Countryman line as witnessed by $(D_n)_{n<\omega}$ (i.e., C is a linear order on ω_1, each D_n is a chain in $C \times C$, and $C \times C = \bigcup_{n<\omega} D_n$). Then from $p = (C, (D_n)_{n<\omega})$ one can define a ladder system on ω_1.

Proof: Note that $L(p) \models \mathrm{ZFC}$ (alternatively, from p we can define $B \subseteq \omega_1$ coding p and work with $L[B]$ instead of $L(p)$). In $L(p)$, C is a linear order on $\kappa = \omega_1^V$ whose square is a countable union of chains. But then $\kappa = \omega_1^{L[p]}$ since for any infinite cardinal κ, no linear order R on κ^{++} can be such that $R \times R$ is the union of $\leq\kappa$–many chains. But now, the $<_{L(p)}$–first ladder system on $\omega_1^{L(p)}$ in $L(p)$ is a ladder system on ω_1 in V. $\square$

Another example:

If $\vec{r} = (r_\nu)_{\nu<\omega_1}$ is an ω_1–sequence of distinct reals, then from $\vec{r}$ one can define a partition $(S_n)_{n<\omega}$ of ω_1 into stationary sets.

On the other hand:

It is consistent that there is a partition of ω_1 into $\aleph_1$–many stationary sets such that no ω_1–sequence of distinct reals can be defined from it:

Proof: Let $\kappa > 2^{\aleph_0}$ be a regular cardinal. Let $\vec{S} = (S_\nu)_{\nu<\kappa}$ be a partition of κ into stationary sets. Let G be generic for the Levy collapse turning κ into ω_1. By κ–c.c. of the Levy collapse, $\vec{S}$ remains a partition of $\omega_1 = \kappa$ into stationary sets. But in $V[G]$ there is no ω_1–sequence of distinct reals $\vec{r}$ definable from $\vec{S}$. Otherwise by the homogeneity of the Levy collapse $\vec{r}$ would be in V. But $\kappa > 2^{\aleph_0}$. $\square$

Also: It is consistent that there is an ω_1–sequence of distinct reals such that no Aronszjan tree is definable from it. In particular, no ladder system on ω_1 is definable from it.

Proof: Suppose there is no ω_2–Aronszajn tree (for example under PFA, but can be forced from just a weakly compact cardinal (Mitchell)). Then $2^{\aleph_0} > \aleph_1$, so there is an ω_2–sequence $\vec{r} = (r_\alpha)_{\alpha<\omega_2}$ of distinct reals. Then after collapsing ω_1 with finite conditions there is no Aronsjan tree T definable from $\vec{r}$. Otherwise, by the homogeneity of the collapse T would be in V and would be an ω_2–Aronszajn tree there. $\square$

Similarly, it is consistent that there is an (ω_1, ω_1)–gap such that no Aronszajn tree is definable from it.

These observations can often be turned into independence results over ZF and implications in ZF. Examples:

Con(ZFC + There is a weakly compact cardinal) implies Con(ZF + There is an (ω_1, ω_1)–gap but no Aronszajn tree).

Con(ZFC) implies Con(ZF + There is a partition of ω_1 into $\aleph_1$–many stationary sets but no ω_1–sequence of distinct reals).

(ZF) The following are equivalent:

- There is a ladder system on ω_1.
- There is a special Aronszajn tree.
- There is a Countryman line.
- There is a Hausdorff gap.
- There is a simplified $(\omega, 1)$–morass.

However:

Question: (ZF) Suppose there is an ω_1–sequence of distinct reals. Does it follow that there is a stationary and co-stationary subset of ω_1?

Definability distinctions and bounded forcing axioms

BPFA: For every proper poset $\mathbb{P}$, $H(\omega_2)^V \preccurlyeq_{\Sigma_1} H(\omega_2)^{V^{\mathbb{P}}}$.

BMM: For every poset $\mathbb{P}$ preserving stationary subsets of ω_1, $H(\omega_2)^V \preccurlyeq_{\Sigma_1} H(\omega_2)^{V^{\mathbb{P}}}$.

Theorem (A.): Suppose BMM holds and $\vec{r}$ is an ω_1–enumeration of distinct reals. Then there is a scale $(s_\alpha)_{\alpha<\omega_2}$ in $({}^\omega\omega, <^*)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{r}$.

Theorem (Todorčević): Suppose BMM holds and $\vec{r}$ is an ω_1–enumeration of distinct reals. Then there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{r}$.

Theorem (Moore): Suppose BPFA holds and $\vec{C}$ is a ladder system on ω_1. Then there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{C}$.

Question: Suppose BPFA holds and $\vec{r}$ is an ω_1–enumeration of distinct reals. Is there a scale $(s_\alpha)_{\alpha<\omega_2}$ in $({}^\omega\omega, <^*)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{r}$? Is there even a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{r}$?

Theorem (A.): Suppose BMM holds and $\vec{r}$ is an ω_1–enumeration of distinct reals. Then there is a scale $(s_\alpha)_{\alpha<\omega_2}$ in $({}^\omega\omega, <^*)$ definable over $\langle H(\omega_2), \in\rangle$ from $\vec{r}$.

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Weaker parameters?

Is there a forcing axiom implying existence of a well–order of $H(\omega_2)$ definable from any partition of ω_1 into $\aleph_1$–many stationary sets?

I don't know this, but

Observation (A.): Suppose BMM holds and the nonstationary ideal on ω_1 is saturated. Let $S \subseteq \omega_1$ be stationary and co-stationary. Then for every $A \in H(\omega_2)$ there is some $r \in \mathbb{R}$ such that $A \in L[S, r]$. Hence, there is some $r \in \mathbb{R}$ and some well–order $\leq$ of $H(\omega_2)$ such that $\leq$ is definable over $\langle H(\omega_2), \in\rangle$ from S and r.

No reasonable forcing axiom is known to imply the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from no parameters (or even from a real number as parameter).

What about proving at least that some (strong) forcing axiom is compatible with the existence of a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from no parameters (or even from a real number as parameter)?

Theorem (A.) If there is a supercompact cardinal, there is a semiproper poset forcing:

(i) PFA^{++} (i.e., for every proper poset $\mathcal{Q}$, every collection D_i ($i < \omega_1$) of dense subsets of ω_1 and every collection $\dot{S}_\alpha$ ($\alpha < \omega_1$) of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\dot{S}_\alpha^G := \{\nu < \omega_1 \,:\, (\exists p \in G)(p \Vdash_{\mathcal{Q}} \nu \in \dot{S}_\alpha)\}$ is stationary for each α).

(ii) There is a well order of $H(\omega_2)$ definable over $H(\omega_2)$ from no parameters.

Theorem (P. Larson) If there is a supercompact limit of supercompact cardinals, there is a semiproper poset forcing:

(i) $\mathsf{MM}^{+\omega}$ (i.e., for every poset $\mathcal{Q}$ preserving stationary subsets of ω_1, every collection D_i ($i < \omega_1$) of dense subsets of ω_1 and every collection $\dot{S}_n$ ($n < \omega$) of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\dot{S}_n^G := \{\nu < \omega_1 \,:\, (\exists p \in G)(p \Vdash_{\mathcal{Q}} \nu \in \dot{S}_n)\}$ is stationary for each n).

(ii) There is a well order of $H(\omega_2)$ definable over $H(\omega_2)$ from no parameters.

MM^{++} fails in Larson's model.

Question: Is MM^{++} consistent with the existence of a well–order of $H(\omega_2)$ lightface definable over $\langle H(\omega_2), \in\rangle$?

Observation: If MM holds and there is no well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ from a real, then there is an ω–club $C \subseteq \omega_2 \cap \mathrm{cf}(\omega)$ such that for every $r \in \mathbb{R}$ there is $\alpha < \omega_2$ such that

$$\langle H(\omega_2), \in\rangle \models \varphi(r, \xi) \leftrightarrow \varphi(r, \xi')$$

for every formula $\varphi(x_0, x_1)$ and for all ξ, ξ' in $C \setminus \alpha$.

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for every formula $\varphi(x_0, x_1)$ and for all ξ, ξ' in $C \setminus \alpha$.

This follows from the following consequences of MM:

(1) If $S \subseteq \omega_1$ is stationary and co-stationary and $X \subseteq \omega_1$, there is $r \in \mathbb{R}$ such that $X \in L[r, S]$. Hence, there is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in \rangle$ from S and some $r \in \mathbb{R}$. (already mentioned)

(2) If $A \subseteq \omega_2 \cap \mathrm{cf}(\omega)$ is stationary and $\omega_2 \cap \mathrm{cf}(\omega)) \setminus A$ is also stationary, then there is some $\alpha < \omega_2$ such that $A \cap \alpha$ and $\alpha \setminus A$ are both stationary.

(3) $u_2 = \omega_2$. Hence, for every $\alpha < \omega_2$ there is a surjection $\pi : \omega_1 \longrightarrow \alpha$ definable over $\langle H(\omega_2), \in \rangle$ from a real.

In particular, if MM holds and there is $A \subseteq \omega_2 \cap \mathrm{cf}(\omega)$ stationary such that $(\omega_2 \cap \mathrm{cf}(\omega)) \setminus A$ is also stationary and A is definable over $H(\omega_2)$ from a real, then there is a well–order of $H(\omega_2)$ definable over $H(\omega_2)$ from a real. Iterating the negation of this implication, using $2^{\aleph_0} = \aleph_2$ and taking diagonal intersections of the corresponding ω–clubs we obtain the observation.

An approach for finding $A \subseteq \omega_2 \cap \mathrm{cf}(\omega)$ stationary with $(\omega_2 \cap \mathrm{cf}(\omega)) \setminus A$ also stationary and A definable over $H(\omega_2)$ (from no parameters):

If MM^{++} holds, then the set A of $\alpha \in \omega_2 \cap \mathrm{cf}(\omega)$ for which there is a transitive $N \models \mathrm{ZFC}^*$ such that

(a) N is correct about stationary subsets of ω_1,

(b) $\omega_2^N = \alpha$, and

(c) there is a filter $G \subseteq \mathrm{Namba}^N$ generic over N

is stationary.

Is $(\omega_2 \cap \mathrm{cf}(\omega)) \setminus A$ also stationary?

Observation (A.): MM^{++} implies that there is an ω–club of $\alpha \in \omega_2 \cap \mathrm{cf}(\omega)$ for which there is a transitive $N \models \mathrm{ZFC}^*$ such that

(a) N is correct about stationary subsets of ω_1,

(b) $\omega_2^N = \alpha$, and

(c) there is a filter $G \subseteq \mathrm{Namba}^N$ generic over N.

And the same holds for any other formula which, provably in ZFC, defines a poset preserving stationary subsets of ω_1 and forcing $\mathrm{cf}(\omega_2^V) = \omega$ (in place of the formula defining Namba forcing).

Proof: Since $u_2 = \omega_2$, $\omega_2^V = \omega_2^{L(\mathbb{R})} = \omega_2^{L(\mathbb{R})^{\mathbb{P}_{max}}}$. But now, given $\alpha \in \omega_2 \cap \mathrm{cf}(\omega)$,

“there is some N such that (a)–(c)”

holds in V if and only if it holds in $L(\mathbb{R})^{\mathbb{P}_{max}}$ by the Π_2 maximality of the $\mathbb{P}_{max}$ extension of $L(\mathbb{R})$, since “...” is Σ_1–expressible in $\langle H(\omega_2), \in, NS_{\omega_1}, r\rangle$ for some real r coding α. So, since $\mathbb{P}_{max} \in L(\mathbb{R})$ is homogeneous, A and $(\omega_2 \cap \mathrm{cf}(\omega)) \setminus A$ are both in $L(\mathbb{R})$. Since A is stationary it has to contain an ω–club: $L(\mathbb{R}) \models$ AD and AD implies that club filter restricted to $\omega_2 \cap \mathrm{cf}(\omega)$ is an ultrafilter.

□

Proof: Since $u_2 = \omega_2$, $\omega_2^V = \omega_2^{L(\mathbb{R})} = \omega_2^{L(\mathbb{R})^{\mathbb{P}_{max}}}$. But now, given $\alpha \in \omega_2 \cap \mathrm{cf}(\omega)$,

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$\square$

Corollary: If there is a supercompact cardinal, then there is an ω–club $C \subseteq u_2$ such that for every $\alpha \in C$ there is a transitive $N \models \text{ZFC}^*$ such that

(a) N is correct about stationary subsets of ω_1,

(b) $\omega_2^N = \alpha$, and

(c) there is a filter $G \subseteq \text{Namba}^N$ generic over N.

And the same holds for any other formula which, provably in ZFC, defines a poset preserving stationary subsets of ω_1 and forcing $\text{cf}(\omega_2^V) = \omega$ (in place of the formula defining Namba forcing).

Conjecture: MM^{++} is incompatible with a well–order of $H(\omega_2)$ definable over $H(\omega_2)$ from a real. In fact, MM^{++} (plus large cardinals) should imply Woodin's $\mathbb{P}_{max}$ axiom $(*)$.

Recently, M. Viale has defined a strong form MM^{+++} of MM^{++} for which he can prove invariance of the theory of $H(\omega_2)$ under forcings preserving stationary subsets of ω_1 and forcing MM^{+++}. MM^{+++} *should* rule out lightface definable well–orders of $H(\omega_2)$.If this is true, then of course it would follow in particular that no forcing axiom for $H(\omega_2)$ would imply the existence of a lightface definable well–order of $H(\omega_2)$.

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A general framework for forcing lightface definable well–orders

General idea:

- Fix a "decoding device" $\mathcal{D} \in H(\kappa^+)$. For example a κ–sequence of pairwise disjoint stationary subsets of κ, or a club–sequence on κ, etc.
- Build a forcing iteration $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, where $\lambda = \kappa^+$ or where λ is some large cardinal (e.g. a supercompact), turning λ into κ^+.
  - (A) Along the iteration, make sure that every subset $X \subseteq \kappa$ gets coded by some ordinal α, $\kappa < \alpha < \lambda$, using $\mathcal{D}$, as witnessed by some club $C \subseteq [\alpha]^{<\kappa}$. (note: $|\alpha| = \kappa$). $\mathcal{W} = \{(X, Y) :$ the first ordinal coding X is less than the first ordinal coding $Y\}$ will be the desired well–order.
  - (B) Make sure that $\mathcal{D}$ becomes definable in the end as the unique object having some property P. Force all necessary objects along the iteration for this to be the case.

Examples: Some positive results with $2^\kappa = \kappa^+$.

This is the philosophy behind the following results.

Theorem (A.) If $\kappa \geq \omega_2$, $\kappa^{<\kappa} = \kappa$, and $2^\kappa = \kappa^+$, then there is a κ–distributive cardinal preserving poset forcing that there is a well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in \rangle$ from no parameters.

Theorem (A.– S. Friedman) If GCH holds, then there is a class–forcing $\mathcal{P}$ such that

(i) $\mathcal{P}$ forces, for every regular $\kappa \geq \omega_2$, that there is a well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in \rangle$ from no parameters, and

(ii) $\mathcal{P}$ preserves "κ is λ–supercompact" for any regular $\kappa \leq \lambda$, hugeness, and other large cardinal notions.

Theorem (A.) If there is a supercompact cardinal, there is a semiproper poset forcing:

(i) PFA^{++} (i.e., for every proper poset $\mathcal{Q}$, every collection D_i ($i < \omega_1$) of dense subsets of ω_1 and every collection $\dot{S}_\alpha$ ($\alpha < \omega_1$) of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\dot{S}_\alpha^G := \{\nu < \omega_1 \ : \ (\exists p \in G)(p \Vdash_{\mathcal{Q}} \nu \in \dot{S}_\alpha)\}$ is stationary for each α).

(ii) There is a well order of $H(\omega_2)$ definable over $H(\omega_2)$ from no parameters.

Theorem (P. Larson) If there is a supercompact limit of supercompact cardinals, there is a semiproper poset forcing:

(i) $\mathrm{MM}^{+\omega}$ (i.e., for every poset $\mathcal{Q}$ preserving stationary subsets of ω_1, every collection D_i ($i < \omega_1$) of dense subsets of ω_1 and every collection $\dot{S}_n$ ($n < \omega$) of $\mathcal{Q}$–names for stationary subsets of ω_1 there is a filter $G \subseteq \mathcal{Q}$ such that $G \cap D_i \neq \emptyset$ for all $i < \omega_1$ and $\dot{S}_n^G := \{\nu < \omega_1 \ : \ (\exists p \in G)(p \Vdash_{\mathcal{Q}} \nu \in \dot{S}_n)\}$ is stationary for each n).

(ii) There is a well order of $H(\omega_2)$ definable over $H(\omega_2)$ from no parameters.

Forcing definable well–orders of $H(\kappa^+)$ together with $2^\kappa > \kappa^+$.

It is possible to force over "small" models so as to obtain a well–order of $H(\kappa^+)$ definable over $H(\kappa^+)$, together with $2^\kappa > \kappa^+$. The first result is probably:

Theorem 5 (Harrington, 1977) Suppose $\omega_1 = \omega_1^L$. Then there is a forcing extension in which $2^{\aleph_0}$ is as large as we want and there is a Δ^1_3 well–order of the reals.

A different approach to forcing lightface definable well–orders

Suppose we want to force a well–order of $H(\kappa^+)$, definable over $H(\kappa^+)$, together with $2^\kappa > \kappa^+$, in a general context (i.e., no anti–large cardinal assumption like $\omega_1 = \omega_1^L$). Of course $\kappa \geq \omega_1$. Recall our initial approach:

- Fix a "decoding device" $\mathcal{D} \in H(\kappa^+)$. For example a κ–sequence of pairwise disjoint stationary subsets of κ, or a club–sequence on κ, etc.
- Build a forcing iteration $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$, where $\lambda = \kappa^+$ or where λ is some large cardinal (e.g. a supercompact), turning λ into κ^+.
  - (A) Along the iteration, make sure that every subset $X \subseteq \kappa$ gets coded by some ordinal α, $\kappa < \alpha < \lambda$, using $\mathcal{D}$, as witnessed by some club $C \subseteq [\alpha]^{<\kappa}$. (note: $|\alpha| = \kappa$).
  - (B) Make sure that $\mathcal{D}$ becomes definable in the end as the unique object having some property P. Force all necessary objects along the iteration for this to be the case.

This approach will not work: Both in (A) and in (B) we have to deal with more than κ^+–many tasks. But we only have κ^+–many ordinals to fulfill task A.

New (naive) ideology: Fix $W \subseteq \mathcal{P}(\kappa)$ of size $\lambda = 2^\kappa$ coding a well–order of $H(\kappa^+)^V$. Code W by a subset A of κ in a first stage (there are ways to do this). Then make A definable in a nice (κ^+–c.c.) iteration of length λ which can be coded as a subset of $H(\kappa^+)$ (there are also ways to do this). In the final extension $V[G]$ well–order the new subsets X of κ by looking at the W–first name $\dot{X} \in H(\kappa^+)$ such that $\dot{X}_G = X$. This procedure is definable from W, but W is definable from A, and A is lightface definable. So we are done. Right?

No: This procedure is definable from W **and** G. So we better make A define **also** G.

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No: This procedure is definable from W **and** G. So we better make A define **also** G.

But A was added in the first stage of the iteration and (most) of G came afterwards!

Solution: Add W also generically along the iterations, in λ–many stages, making sure that it codes the relevant parts of G.

So the iteration should look like:

(1) At stage 0, Force $A \subseteq \kappa$ coding a given $W \subseteq {}^{\kappa}\kappa$ of size λ coding a well-order of $\mathcal{P}(\kappa)$.
(2) Along the iteration, aim at making A definable. In the end it should become the unique object having a certain property P.
(3) Along the iteration build an increasing chain W_α of predicates extending W, $W_\alpha \subseteq {}^{\kappa}\kappa$. These predicates code longer and longer fragments of the generic G.
(4) Argue that A actually codes all future W_α in the intended way.

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Theorem 6 (A.–Holy–Lücke, 2013) Let $\kappa \geq \omega_1$ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Then there is a partial order with the following properties.

(i) $\mathcal{P}$ is $<\kappa$–closed.

(ii) $\mathcal{P}$ has the κ^+–chain condition.

(iii) $|\mathcal{P}| = 2^\kappa$

(iv) $\mathcal{P}$ forces the existence of a well–order $\mathcal{W}$ of $H(\kappa^+)$ such that $\mathcal{W}$ is definable over $\langle H(\kappa^+), \in\rangle$ without parameters.

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So, if $2^\kappa = \lambda$ in V, then $2^\kappa = \lambda$ holds in the extension.

$\mathcal{P}$ will be $\mathcal{P}_\lambda$ for a certain iteration $\langle \mathcal{P}_\alpha \mid \alpha \leq \lambda \rangle$. Each $\mathcal{P}_\alpha$ will be a subset of $H(\kappa^+)$.

Next: I will present the some of the main ingredients used in the construction, then I will give a rough overview of the construction.

Almost disjoint coding with clubs

This is a variation of Solovay's *almost disjoint coding forcing*.

Solovay's almost disjoint coding forcing: Suppose $\kappa^{<\kappa} = \kappa$ and $(s_\alpha : \alpha < \kappa)$ enumerates ${}^{<\kappa}\kappa$. Let $W \subseteq {}^{\kappa}\kappa$. Then there is a $<\kappa$–closed forcing Q_0^W with the κ^+–chain condition and forcing

$$f : \kappa \longrightarrow 2$$

such that for every $g \in ({}^{\kappa}\kappa)^V$, $g \in W$ if and only if there is some $\alpha < \kappa$ such that for all $\beta > \alpha$, if $s_\beta \subseteq g$, then $f(\beta) = 0$:

$(t, X) \in Q_0^W$ iff $t \in {}^{<\kappa}2$ and $X \in [W]^{<\kappa}$.
Given (t_0, X_0), (t_1, X_1) in Q_0^W, $(t_1, X_1) \leq_{Q_0^W} (t_0, X_0)$ iff

(a) $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$, and

(b) for every $\alpha \in dom(t_1) \setminus dom(t_0)$ and every $h \in X_0$, if $s_\alpha \subseteq h$, then $t_1(\alpha) = 0$.

We don't have a definable way to distinguish between $g \in ({}^{\kappa}\kappa)^V$ and $g \in ({}^{\kappa}\kappa)^{V[G]} \setminus ({}^{\kappa}\kappa)^V$ in the extension. But we would like W to be the collection of **all** $g \in ({}^{\kappa}\kappa)^{V[G]}$ such that there is some $\alpha < \kappa$ such that for all $\beta > \alpha$, if $s_\beta \subseteq g$, then $f(\beta) = 0$. This will make W indeed definable from f *and* $(s_\alpha : \alpha < \kappa)$.

A solution for this: Add $(s_\alpha : \alpha < \kappa)$ generically together with f. Call this forcing Q_1^W:

$(\vec{s}, t, X) \in Q_1^W$ iff $\vec{s} : \gamma \longrightarrow {}^{<\kappa}\kappa$ with $\gamma < \kappa$ a successor ordinal, $t : \gamma \longrightarrow 2$, and $X \in [W]^{<\kappa}$.
Given $(\vec{s}_0, t_0, X_0)$, $(\vec{s}_1, t_1, X_1)$ in Q_1^W, $(\vec{s}_1, t_1, X_1) \leq_{Q_1^W} (s_0, t_0, X_0)$ iff

(a) $\vec{s}_0 \subseteq \vec{s}_1$,
(b) $t_0 \subseteq t_1$ and $X_0 \subseteq X_1$, and
(c) for every $\alpha \in dom(t_1) \setminus dom(t_0)$ and every $h \in X_0$, if $s_\alpha \subseteq h$, then $t_1(\alpha) = 0$.

This is not good enough for our purposes, though:

We want to build an iteration. However, if $W_\alpha \subseteq W_{\alpha'}$ as in task (3) in our scheme

“Along the iteration build an increasing chain W_α of predicates extending W, $W_\alpha \subseteq H(\kappa^+)$. These predicates code longer and longer fragments of the generic G”,

then $Q_1^{W_\alpha}$ is **not** a complete suborder of $Q_1^{W_{\alpha'}}$.

The solution is to do the coding relative to generic clubs. Call the resulting forcing Q_2^W:

$(\vec{s}, t, (c_x : x \in X)) \in Q_2^W$ iff

(i) $\vec{s} : \gamma \longrightarrow {}^{<\kappa}\kappa$ with $\gamma < \kappa$ a successor ordinal,

(ii) $t : dom(\vec{s}) \longrightarrow 2$,

(iii) $X \in [W]^{<\kappa}$, and

(iv) for every $x \in X$, c_x is a closed subset of γ and

$$s_\alpha \subseteq x \to t(\alpha) = 0$$

for all $\alpha \in c_x$.

Given $(\vec{s}_0, t_0, (c_x^0 : x \in X_0))$, $(\vec{s}_1, t_1, (c_x^1 : x \in X_1))$ in Q_2^W, $(\vec{s}_1, t_1, (c_x^1 : x \in X_1)) \leq_{Q_2^W} (s_0, t_0, (c_x^0 : x \in X_0))$ iff

(a) $\vec{s}_0 \subseteq \vec{s}_1$,

(b) $t_0 \subseteq t_1$, and $X_0 \subseteq X_1$, and

(c) for every $x \in X_0$, $c_x^0 = c_x^1 \cap dom(\vec{s}_0)$.

Lemma: If $W \subseteq {}^{\kappa}\kappa$, then

(1) Q_2^W is $<\kappa$–closed,

(2) Q_2^W has the κ^+–c.c., and

(3) if $\vec{s} = (s_\alpha : \alpha < \kappa)$ and $T : \kappa \longrightarrow 2$ are given by the generic filter, then for every $x \in {}^{\kappa}\kappa$ in the extension, $x \in W$ if and only if there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$,

$$s_\alpha \subseteq x \to T(\alpha) = 0$$

(4) Furthermore, if $W \subseteq W'$, then Q_2^W is a complete suborder of $Q_2^{W'}$.

Another ingredient: Strongly type-guessing sequences

This appears already in the proofs of the following Theorems:

Theorem (A.) If $\kappa \geq \omega_2$, $\kappa^{<\kappa} = \kappa$, and $2^\kappa = \kappa^+$, then there is a κ–distributive cardinal preserving poset forcing that there is a well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in \rangle$ from no parameters.

Theorem (A– S. Friedman) If GCH holds, then there is a class–forcing $\mathcal{P}$ such that

(i) $\mathcal{P}$ forces, for every regular $\kappa \geq \omega_2$, that there is a well–order of $H(\kappa^+)$ definable over $\langle H(\kappa^+), \in \rangle$ from no parameters, and

(ii) $\mathcal{P}$ preserves "κ is λ–supercompact" for any regular $\kappa \leq \lambda$, hugeness, and other large cardinal notions.

Given two sets of ordinals X and Y, let $X \cap^* Y$ be the collection of all $\delta \in X \cap Y$ such that δ is not a limit point of X. ($\cap^*$ is not commutative: $\{\omega\} \cap^* (\omega+1) = \{\omega\}$ but $(\omega+1) \cap^* \{\omega\} = \emptyset$.)

$\vec{C} = (C_\delta \mid \delta \in \mathrm{dom}\,(\vec{C}))$ is a club–sequence if $\mathrm{dom}(\vec{C})$ is a set of ordinals and C_δ is a club subset of δ for each $\delta \in \mathrm{dom}(\vec{C})$.

$\vec{C}$ is *coherent* if there is a club sequence $\vec{D} = (D_\delta \mid \delta \in \mathrm{dom}\,(\vec{D}))$ such that

- $\vec{C} \subseteq \vec{D}$ and
- for every $\delta \in \mathrm{dom}(\vec{D})$ and every limit point γ of D_δ, $\gamma \in \mathrm{dom}(\vec{D})$ and $D_\gamma = D_\delta \cap \gamma$.

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$\tau \in \mathrm{Ord}$ is the *height of* $\vec{C}$, $\mathrm{hgt}(\vec{C}) = \tau$, if $\mathrm{ot}(C_\delta) = \tau$ for all $\delta \in \mathrm{dom}(\vec{C})$.

A club–sequence $\vec{C} = (C_\delta \mid \delta \in \mathrm{dom}\,(\vec{C}))$ with stationary domain such that $\sup(\mathrm{dom}(\vec{C})) = \chi$ is *strongly type-guessing* if for every club subset $C \subseteq \chi$ there is a club $D \subseteq \chi$ such that $\mathrm{ot}(C_\delta \cap^* C) = \mathrm{ot}(C_\delta)$ for every $\delta \in \mathrm{dom}(\vec{C}) \cap D$.

Given a set of ordinals X and on ordinal η, define $\mathrm{rank}_X(\eta) \geq \mu$ by recursion as follows:

- $\mathrm{rank}_X(\eta) > 0$ if and only if there is a nonempty $X' \subseteq X$ such that $\sup(X') = \eta$.
- If $\mu > 0$, then $\mathrm{rank}_X(\eta) > \mu$ if and only if η is a limit of ordinals ξ such that $\mathrm{rank}_X(\xi) \geq \mu$.

An ordinal η is perfect if $\mathrm{rank}_\eta(\eta) = \eta$.

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Overview of the iteration

Let $(\eta_\alpha \mid \alpha < \kappa)$ be the strictly increasing enumeration of all nonzero perfect ordinals in κ. Let also $S \subseteq \kappa \cap \mathrm{cf}(\omega)$ be stationary and such that $(\kappa \cap \mathrm{cf}(\omega)) \setminus S$ is also stationary.

In our construction, we will start by adding a ladder system $\vec{E}$ on S and also $\vec{s} = (s_\xi \mid \xi < \kappa)$ and T as in $Q_1^{W_0}$. W_0 will be a certain subset of ${}^\kappa\kappa$ in V coding a well–order of $H(\kappa^+)$.

Then we will add, by initial segments, a sequence $(\vec{C}^\alpha \mid \alpha < \kappa)$ such that

(a) each $\vec{C}^\alpha$ is a coherent club–sequence with $\mathrm{dom}(\vec{C}^\alpha) \subseteq \kappa \setminus S$,

(b) $\mathrm{dom}(\vec{C}^\alpha) \cap \mathrm{dom}(\vec{C}^{\alpha'}) = \emptyset$ for all $\alpha \neq \alpha'$, and

(c) $\{\xi < \kappa : (\exists\alpha)(\mathrm{hgt}(\vec{C}^\alpha) = \eta_\xi)\}$ codes $\vec{s}$, T and $\vec{E}$.

From then on, we iterate in length λ. At any given stage β we do the following:

(1) We are presented with a club $C \subseteq \kappa$ (by a book-keeping given by W_0) and we add by initial segments a club $D_\beta \subseteq \kappa$ such that for all α and all $\delta \in D_\beta \cap \mathrm{dom}(\vec{C}^\alpha)$,

$$\mathrm{ot}(C^\alpha_\delta \cap^* C) = \mathrm{hgt}(\vec{C}^\alpha)$$

(2) We define a certain extension $W_\beta \subseteq {}^\kappa\kappa$ of the previous $W_{\beta'}$ $(\beta' < \beta)$ coding more information of the generic object.

(3) We add clubs $D \subseteq \kappa$ witnessing that T codes W_β relative to $\vec{s} = (s_\xi \mid \xi < \kappa)$ *and* relative to S (this means that the almost disjoint coding condition takes place on all $\alpha \in D \cap S$ rather than $\alpha \in D$).

(4) We also add clubs witnessing that $\vec{E}$ becomes strongly club–guessing.

We define everything in a neat way so that $\mathcal{P}_\alpha \subseteq H(\kappa^+)$ and that the generic G_λ can be decoded from W_λ together with $\vec{s}$, T and $\vec{E}$, and therefore from $\vec{s}$, T and $\vec{E}$.

In the end, every $\vec{C}^\alpha$ is a strongly type-guessing club–sequence with stationary domain disjoint from S. [Easy, by construction.] Also, the construction guarantees, by density arguments, that the class of S in $\mathcal{P}(\kappa)/NS_\kappa$ is the largest class such that all its members carry a strongly guessing club–sequence, and that if $\eta \notin \{\text{hgt}(\vec{C}^\alpha) \mid \alpha < \kappa\}$ and $\vec{C}$ is a coherent club–sequence with stationary domain disjoint from S and height η, then for a tail of β,

$$\{\delta \in \text{dom}(\vec{C}) \mid \text{ot}(C_\delta \cap^* D_\beta) < \eta\}$$

is stationary.

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$$\{\delta \in \mathrm{dom}(\vec{C}) \mid \mathrm{ot}(C_\delta \cap^* D_\beta) < \eta\}$$

is stationary.

Hence $(\vec{s}, T, \vec{E})$ is coded by the set of $\xi < \kappa$ such that there $\vec{C}$ such that

- $\vec{C}$ is a strongly type-guessing club–sequence with stationary domain disjoint from S', where the class of S' in $\mathcal{P}(\kappa)/NS_\kappa$ is maximal such that all its members carry a strongly guessing club–sequence, and
- $\vec{C}$ has height η_ξ.

By the chain condition of $\mathcal{P}_\lambda$ there is a well–order $\mathcal{W}^*$ of $H(\kappa^+)^{V[G]}$ definable from W_0, G and $\vec{E}$,

hence from W_λ, T, $\vec{E}$, and $\vec{s}$,

hence from T,

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□

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In the proof of this Theorem, $\kappa^{<\kappa} = \kappa$ is heavily needed to get κ^+–c.c.

How about dropping the requirement $\kappa^{<\kappa} = \kappa$? For example:

Question: Is $2^{\aleph_0} > \aleph_1$ + $2^{\aleph_1} > \aleph_2$ + "There is a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ without parameters" compatible with large cardinals?

Thank you for your attention!