Recall:

We defined \( (V_\alpha : \alpha \in \text{Ord}) \) by recursion on \( \text{Ord} \):

- \( V_0 = \emptyset \)
- \( V_{\alpha+1} = \mathcal{P}(V_\alpha) \)
- \( V_\delta = \bigcup \{ V_\beta : \beta < \delta \} \) if \( \delta \) is a limit ordinal.

\( (V_\alpha : \alpha \in \text{Ord}) \) is called the cumulative hierarchy.

**Definition**

For every \( x \in \bigcup_{\alpha \in \text{Ord}} V_\alpha \),

\[ \text{rank}(x) = \min \{ \alpha \in \text{Ord} : x \in V_{\alpha+1} \} \]
We have seen, by induction on the ordinals, that $V_\alpha$ is transitive for every ordinal $\alpha$.

**Proposition**

*For all $\alpha < \beta$, $V_\alpha \subseteq V_\beta$.***

**Proof.**

Again by induction on $\beta$. This is vacuously true for $\beta = 0$. For $\beta$ a nonzero limit ordinal, $V_\beta \supseteq V_\alpha$ by definition of $V_\beta$. For $\beta = \bar{\beta} + 1$, $V_\beta = \mathcal{P}(V_{\bar{\beta}})$. If $\alpha = \bar{\beta}$, then we are done since every member of $V_{\bar{\beta}}$ is a subset of $V_{\bar{\beta}}$ (as $V_{\bar{\beta}}$ is transitive) and therefore $V_{\bar{\beta}} \subseteq \mathcal{P}(V_{\bar{\beta}})$. If $\alpha < \bar{\beta}$, then $V_\alpha \subseteq V_{\bar{\beta}}$ by induction hypothesis. But $V_{\bar{\beta}} \subseteq \mathcal{P}(V_{\bar{\beta}})$ by the previous case, and hence $V_\alpha \subseteq \mathcal{P}(V_{\bar{\beta}}) = V_\beta$. \qed
Definition
For every \( x \in \bigcup_{\alpha \in \text{Ord}} V_{\alpha} \), let \( \text{rank}(x) \) be the first \( \alpha \) such that \( x \in V_{\alpha+1} \).

Definition
For every set \( x \), the transitive closure of \( x \), denoted by \( \text{TC}(x) \), is \( \bigcup \{ X_n : n < \omega \} \) where

- \( X_0 = x \)
- \( X_{n+1} = \bigcup X_n \)

So \( \text{TC}(x) = x \cup \bigcup x \cup \bigcup \bigcup x \cup \bigcup \bigcup \bigcup x \cup \ldots \)

[Exercise: \( \text{TC}(x) \) is the \( \subseteq \)-least transitive set \( y \) such that \( x \subseteq y \). In other words, \( \text{TC}(x) = \bigcap \{ y : y \text{ transitive}, x \subseteq y \} \).]
Definition

$\forall$ denotes the class of all sets; that is, $\forall = \{x : x = x\}$.

Definition

$WF = \bigcup\{V_\alpha : \alpha \in \text{Ord}\}$: The class of all $x$ such that $x \in V_\alpha$ for some ordinal $\alpha$.

Note: $WF$ is a transitive class: $y \in x \in V_\alpha$ implies $y \in V_\alpha$ since $V_\alpha$ is transitive.
Theorem
(ZF) $\forall = WF$

Proof: Suppose, towards a contradiction, that there is some set $x$ such that $x \not\in WF$. Let $y = TC(x)$.

$y \not\in WF$: Suppose $y \in V_\alpha$. Since $x \subseteq y \subseteq V_\alpha$ (where $y \subseteq V_\alpha$ is true by transitivity of $V_\alpha$), $x \in \mathcal{P}(V_\alpha) = V_{\alpha+1}$. Contradiction.

By Foundation we may find $a \in y \cup \{y\}$, $a \in$–minimal in $y \cup \{y\}$, such that $a \not\in WF$. 
For every $z \in a$, it follows that $z \in y \cup \{y\}$ (by transitivity of $y \cup \{y\}$) and therefore $z \in V_\alpha$ for some $\alpha \in \text{Ord}$ by $\in$–minimality of $a$ among $\{w \in y \cup \{y\} : w \not\in \text{WF}\}$. Hence, the function rank $\upharpoonright a$ sending $z \in a$ to rank$(z)$ is defined for all $z \in a$.

But then range(rank $\upharpoonright a$) has to be a set by Replacement and therefore there is some ordinal $\bar{\alpha}$ such that $\bar{\alpha} > \text{rank}(z)$ for every $z \in a$ [No set $X$ of ordinals can be cofinal in $\text{Ord}$ (i.e., such that for every $\alpha \in \text{Ord}$ there is some $\beta \in X$ with $\alpha < \beta$). Why? Otherwise $\bigcup X = \text{Ord}$, which is not a set (Burali–Forti), but $\bigcup X$ is a set if $X$ is a set by Union Axiom. Contradiction.]

It follows that for every $z \in a$ there is some $\beta < \bar{\alpha}$ such that $z \in V_\beta \subseteq V_{\bar{\alpha}}$.

Hence $a \subseteq V_{\bar{\alpha}}$ and therefore $a \in \mathcal{P}(V_{\bar{\alpha}}) = V_{\bar{\alpha}+1}$. Contradiction with $a \notin \text{WF}$. $\square$
The fact that $\bigvee = \textbf{WF}$ realises the idea that a set is any collection built out of sets already built.

This is known as the *iterative conception* of sets. Note that this conception of sets rules out such “sets” as $\bigvee$ or the Russell class $\{x : x \notin x\}$: They couldn’t possibly be sets since one needs to refer to the totality of sets for their definition, a totality to which they would belong if they were sets. Take for example, $\bigvee$. Certainly, if $\bigvee$ is a set, then $\bigvee \in \bigvee$. But this goes against the iterative conception of set, whereby a set is built up out of previously built sets.
The picture of the universe provided by $\mathcal{V} = \mathbf{WF}$ is a very appealing and very natural one (once one has come across it, at least). This picture of the universe of all sets, and the fact that $\mathbf{ZF}$ implies $\mathcal{V} = \mathbf{WF}$, is the main source of *intrinsic* justifications of the $\mathbf{ZF}$ axioms.
Inner models and relativization

Let \((M, \in^M)\) be a submodel, or inner model, defined by a formula \(\Theta(x)\); in other words, \(M = \{a : \Theta(a)\}\) and, for all \(a, b \in M\), \(a \in^M b\) if and only if \(a \in b\) (we usually leave out \(\in^M\) and write \(M\) instead of \((M, \in^M)\)).

(Examples: \(\forall\), \(WF\), \(L\), \(HOD\), ...).

We define the relativization to \(M\) of a formula \(\varphi(\vec{x})\), to be denoted \(\varphi^M(\vec{x})\), in the following manner.

- \((x \in y)^M\) is \(x \in y\).
- \((x = y)^M\) is \(x = y\).
- \((\varphi_0 \lor \varphi_1)^M\) is \(\varphi_0^M \lor \varphi_1^M\).
- \((\neg \varphi)^M\) is \(\neg \varphi^M\).
- \(((\forall x)(\varphi(\vec{x}))^M\) is \(\forall x(\Theta(x) \rightarrow \varphi^M(\vec{x}))\). We may also write \((\forall x \in M)\varphi^M(x)\).
The following is quite obvious and can be easily proved by induction on the complexity of $\sigma$:

**Note:** Suppose $T$ is a theory in the language of set theory. Suppose $(N, E)$ is a structure in the language of set theory, and suppose $M$ is an inner model in $N$. Then $(N, E) \models \sigma^M$ for every $\sigma \in T$ if and only if $(M, E) \models T$.

**Notation:** If $(N, E)$ is a structure in the language of set theory, $M$ is an inner model defined by a formula $\Theta(x)$ possibly with parameters (i.e., $M = (N, E \upharpoonright (M \times M))$, where $M = \{ a \in N : (N, E) \models \Theta(a) \}$), and we want / need to emphasise that $M$ is the inner model defined by $\Theta(x)$ as defined within $(N, E)$, then we often write $M^N$ instead of $M$.

**Example:** $WF^M$

**Note:** For every ordinal $\alpha$, $V^WF_\alpha = V_\alpha$ (here $V_\alpha$ refers to the set, definable from the parameter $\alpha$, with the definition that we have seen).
Many facts about the universe $\forall$ are inherited by reasonable submodels. For example:

**Lemma**

*Suppose $M$ is a transitive set or a transitive proper class. Then $M \models \text{Axiom of Extensionality}$.***

**Proof:** Let $a, b \in M$ and suppose $M \models (\forall x)(x \in a \leftrightarrow x \in b)$

(this of course is shorthand for

$$M \models (\forall x)(x \in y \leftrightarrow x \in z)[\bar{a}]$$

where $\bar{a}$ is any assignment sending the variable $y$ to $a$ and the variable $z$ to $b$).
This means that \( a \cap M = b \cap M \).

Since \( M \) is transitive (in \( V \)), every member of \( a \) or of \( b \) is a member of \( M \). It follows that \( a \cap M = a \) and \( b \cap M = b \) and therefore \( a = b \). Hence \( M \models a = b \).

In sum, \( M \) thinks that for all \( y, z \), if \( y \) and \( z \) have the same elements, then they are equal. In other words, \( M \models \text{Axiom of Extensionality} \). \( \square \)
Also:

**Lemma**

Suppose $M$ is a transitive set or a transitive proper class which is closed under unordered pairs (meaning that for all $a, b \in M$, \{a, b\} \in M). Then $M \models$ Axiom of Unordered pairs.

**Proof.**

Let $c = \{a, b\} \in M$. Check, as in the previous proof, that $M \models (\forall x)x \in c \iff x = a \lor x = b$. 

□
Similarly:

**Lemma**

Suppose $M$ is a transitive set or a transitive proper class. Suppose $\bigcup a \in M$ for every $a \in M$. Then $M \models \text{Union set Axiom}.$

**Lemma**

Suppose $M$ is a transitive set or a transitive proper class. Suppose for every $a \in M$ there is some $b \in M$ such that $b = \mathcal{P}(a) \cap M$. Then $M \models \text{Power set axiom}.$

[Proofs: Exercises.]

**Note:** There are situations in which there are transitive models $M$ of fragments of ZFC, or even of all of ZFC, and some $a \in M$ such that $\mathcal{P}(a)^M$ is strictly included in $\mathcal{P}(a)$ (i.e., there are subsets $b$ of $a$ such that $b \not\in M$).
Lemma
Suppose $M$ is a transitive set or a transitive proper class. If $\omega \in M$, then $M \models \text{Infinity}$.

Proof idea: As in the previous proofs. The point is that $M$ recognises $\emptyset$ correctly, recognises correctly that something is an ordinal, and recognises correctly that something is the successor of an ordinal.

We say that the notion of ordinal is absolute with respect to transitive models. It is possible to identify large families of properties that are absolute with respect to transitive models by virtue of their being definable by syntactically ‘simple’ formulas (from the point of view of their quantifiers). We don’t need this kind of general analysis at the moment so we won’t go into that now.
Note: The notion of finiteness is also absolute with respect to transitive models but, on the other hand, the notion of countability is highly non–absolute with respect to transitive models:

There are transitive models $M$ and $a \in M$ such that $M \models a$ is uncountable

but there is a bijection $f : \omega \longrightarrow a$, so $a$ is countable in $\mathbb{V}$. The problem of course is that $f$ is not in $M$.

We will soon see that there are transitive models of (fragments of) ZFC such that all their sets are countable in $\mathbb{V}$. And even the whole model can be countable in $\mathbb{V}$. 
**Note:** The notion of finiteness is also absolute with respect to transitive models but, on the other hand, the notion of countability is highly *non–absolute* with respect to transitive models:

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We will soon see that there are transitive models of (fragments of) ZFC such that all their sets are countable in $\mathbb{V}$. And even the whole model can be countable in $\mathbb{V}$. 
The notion of choice function is also absolute with respect to transitive models: If $M$ is transitive, $a \in M$ consists of nonempty sets, $f \in M$, and $f$ is a choice function for $M$, then $M \models \text{“} f \text{ is a choice function for } a \text{”}$. Hence:

**Lemma**

Let $M$ be a transitive set or a transitive proper class. Suppose for every $a \in M$ consisting of nonempty sets there is a choice function $f$ for $a$, $f \in M$. Then $M \models \text{AC}$. 
Lemma
Let $M$ be a transitive set or a transitive proper class. Suppose
$b \in M$ whenever $a \in M$ and $b \subseteq a$ is definable over $M$, possibly
from parameters (in other words, $b = \{ c : c \in a, M \models \varphi(c, \bar{p}) \}$
for some parameters $\bar{p} \in M$). Then $M \models \text{Separation}$.

Lemma
Let $M$ be a transitive set or a transitive proper class. Suppose
$F[a] \in M$ whenever $a \in M$, and $F$ is a class–function over $M$ (in
other words, if $F$ is definable by a formula $\varphi(x, y, \bar{z})$ which, over
$M$ is functional, $\bar{p} \in M$, and $a \in M$, then
$\{ c : (\exists b \in a)M \models \varphi(b, c, \bar{p}) \} \in M$). Then $M \models \text{Replacement}$.
Our first relative consistency proof: 
Con(ZF \{Foundation\}) implies Con(ZF)

**Theorem**

Let $M \models ZF \{Foundation\}$. Then $M \models \sigma^{WF^M}$ for every $\sigma \in ZF$. Hence, $WF^M \models ZF$.

**Proof:** By the previous lemmas and the construction of $(V_\alpha : \alpha \in \text{Ord})$, $WF^M \models \sigma$ for every axiom $\sigma$ of $ZF \{Foundation\}$

[go through them one by one them and check that $WF$ is closed under the relevant operation, then apply the relevant lemma].
To see that $M \models \text{Foundation}^{WF}$ holds, let us work in $M$: Let $a \in V_\alpha$, let $Z \subseteq a$, $Z \in WF$, and let $b \in Z$ such that \( \text{rank}(b) = \min \{ \text{rank}(z) : z \in Z \} \). Then

$$WF \models \text{rank}(b) = \min \{ \text{rank}(z) : z \in Z \}$$

by absoluteness of the relevant notions. Hence, $WF$ thinks that the restriction of $\in$ to $a$ is well–founded. Since this is true for all $a \in WF$,

$$WF \models \text{Foundation}$$

\qed
Corollary

If $ZF \setminus \{\text{Foundation}\}$ is consistent, then $ZF$ is consistent.

Proof.

Suppose $ZF \setminus \{\text{Foundation}\}$. By the completeness theorem we may find a model $M \models ZF \setminus \{\text{Foundation}\}$. Let $M' = WF^M$. By the theorem $M' \models ZF$. Hence, $ZF$ has a model and therefore it is consistent.

\[ \square \]

Remark

By exactly the same argument, if $M \models ZFC \setminus \{\text{Foundation}\}$, then $M \models \sigma^{WF^M}$ for every $\sigma \in ZFC$. Hence, $Con(ZFC \setminus \{\text{Foundation}\})$ implies $Con(ZFC)$. 
Similar relative consistency results:

One can define “the constructible universe” $L$:

- $L_0 = \emptyset$
- $L_{\alpha+1} = \text{Def}(L_\alpha)$, where $\text{Def}(L_\alpha)$ is the set of all subsets of $L_\alpha$ definable over $L_\alpha$ possibly with parameters, i.e., the collection of all sets of the form
  \[
  \{ b \in L_\alpha : L_\alpha \models \varphi(b, a_0, \ldots, a_{n-1}) \}
  \]
  for some formula $\varphi(x, \bar{x})$ and $a_0, \ldots, a_{n-1} \in L_\alpha$.
- $L_\delta = \bigcup_{\alpha < \delta} L_\alpha$ if $\delta > 0$ is a limit ordinal.

$L$ is then $\bigcup_{\alpha \in \text{Ord}} L_\alpha$.

This construction is due to Gödel. He proved that if we do this construction in $ZF$, then $L \models ZF$ but also $L \models AC$ and $L \models CH$. 
The above results imply that if $ZF$ is consistent, then $ZFC$ is also consistent, and in fact also $ZFC+CH$. Linking this to the implication we have seen we thus have that if $ZF \setminus \{\text{Foundation}\}$ is consistent, then so is $ZFC+CH$.

These relative consistency proofs proceed by building suitable inner models.\(^1\)

Most relative consistency proofs proceed, on the other hand, by building suitable outer models of some given ground model. The construction of these outer models is done with the forcing method. This is an extremely powerful method in set theory. I will try to say something more specific about this method later on.

\(^1\)What set–theorists understand by “inner models” are usually much more complicated than $WF$ or $L$. However, the construction of $L$ is in fact the paradigm for most of these more complicated constructions.
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