Adding many Cohen reals

Let $\alpha$ be an ordinal and let $\text{Add}(\omega, \alpha)$ be the following forcing notion. A condition in $\text{Add}(\omega, \alpha)$ is a finite function $p \subseteq (\alpha \times \omega) \times 2$. Given $\text{Add}(\omega, \alpha)$–conditions $p_0, p_1$, $p_1 \leq_{\text{Add}(\omega, \alpha)} p_0$ iff $p_0 \subseteq p_1$.

Lemma

Let $M$ be a transitive model of (enough of) ZF, let $\alpha \in M$ be an ordinal, and let $\beta < \alpha$. If $G$ is generic for $\text{Add}(\omega, \alpha)^M$ ($= \text{Add}(\omega, \alpha)$) over $M$, then

$$c^G_\beta = \{(n, \epsilon) \in \omega \times 2 : ((\alpha, n), \epsilon) \in p \text{ for some } p \in G\}$$

is a Cohen real over $M$ (in other words, $c^G_\beta$ is $2^{<\omega}$–generic over $M$). Furthermore, if $\beta < \beta' < \alpha$, then $c^G_\beta \neq c^G_{\beta'}$. 
As we saw, the proof of the lemma is a simple density argument.

Corollary

If $M$ is a transitive model of enough of ZFC, $\alpha \in M$ is an ordinal, and $G$ is Add($\omega$, $\alpha$)—generic over $M$, then $M[G] \models 2^{\aleph_0} \geq |\alpha|$.

In this corollary, if in particular $\alpha = \aleph_{1000}$, then $M[G] \models 2^{\aleph_0} \geq |\aleph_{1000}^V|$. Is this enough to conclude the consistency of ZFC+$2^{\aleph_0} \geq \aleph_{1000}$ relative to that of ZFC? Not quite yet. For all we know so far it could be that $M[G] \models |\aleph_{1000}^M| < \aleph_{1000}$. It could even be that $\aleph_{1000}^M$ has become countable in $M$. We will next see that this does not happen, and in fact $\aleph_{1000}^M = \aleph_{1000}^{M[G]}$. 
As we saw, the proof of the lemma is a simple density argument.

**Corollary**

If $M$ is a transitive model of enough of ZFC, $\alpha \in M$ is an ordinal, and $G$ is $\text{Add}(\omega, \alpha)$–generic over $M$, then $M[G] \models 2^{\aleph_0} \geq |\alpha|$.

In this corollary, if in particular $\alpha = \aleph_{1000}$, then $M[G] \models 2^{\aleph_0} \geq |\aleph_{1000}^V|$. Is this enough to conclude the consistency of $\text{ZFC} + 2^{\aleph_0} \geq \aleph_{1000}$ relative to that of ZFC? Not quite yet. For all we know so far it could be that $M[G] \models |\aleph_{1000}^M| < \aleph_{1000}$. It could even be that $\aleph_{1000}^M$ has become countable in $M$. We will next see that this does not happen, and in fact $\aleph_{1000}^M = \aleph_{1000}^M[G]$. 
Chain condition and cardinal preservation

Definition
Given a partial order $\mathbb{P}$, $A \subseteq \mathbb{P}$ is an antichain if $p$ and $p'$ are incompatible (i.e., there is no $q \in \mathbb{P}$, $q \leq_p p$, $q \leq_p p'$) for all $p$, $p' \in A$, $p \neq p'$. $A \subseteq \mathbb{P}$ is a maximal antichain if $A$ is an antichain and it is $\subseteq$–maximal with respect to $\subseteq$ (equivalently, for every $p \in \mathbb{P}$ there is some $p' \in A$ and some $q \in \mathbb{P}$ such that $q \leq_p p$, $p'$).

Definition
Given a cardinal $\kappa$, a partial order $\mathbb{P}$ has the $\kappa$–chain condition ($\mathbb{P}$ has the $\kappa$–c.c.) iff there are no antichains $A$ of $\mathbb{P}$ such that $|A| = \kappa$. If $\mathbb{P}$ has the $\aleph_1$–c.c., then we also say that $\mathbb{P}$ has the countable chain condition ($\mathbb{P}$ is c.c.c.).
Fact (ZFC) Suppose $\mathbb{P}$ is a forcing notion and $D \subseteq \mathbb{P}$ is dense. Then there is a maximal antichain $A$ of $\mathbb{P}$ such that $A \subseteq D$.

Proof: We build a sequence $(p_i)_{i < \lambda}$ of pairwise incompatible conditions in $D$.

If at some stage $\lambda_0$ in the construction $\{p_i \mid i < \lambda_0\}$ is a maximal antichain in $\mathbb{P}$, then we are done.

If $\{p_i \mid i < \lambda_0\}$ is not maximal, then using the fact that $D$ is dense we can find some $p_{\lambda_0} \in D$ incompatible with $p_i$ for all $i < \lambda_0$. This specifies stage $\lambda_0 + 1$ of the construction.

The construction stops at some point, thus giving rise to a maximal antichain. $\square$
Lemma
Suppose \( \mathbb{P} \) is a forcing notion, \( \lambda < \kappa \) are infinite cardinals, \( \dot{f} \) is a \( \mathbb{P} \)–name, and \( p \in \mathbb{P} \) is a condition such that

\[
p \Vdash _\mathbb{P} \dot{f} \text{ is a function, } \dot{f} : \check{\lambda} \to \check{\kappa}
\]

Suppose \( \mathbb{P} \) is c.c.c. Then \( p \Vdash _\mathbb{P} \dot{f} \) is not onto \( \check{\kappa} \).

Proof: For every \( \xi \in \lambda \) let

\( D_\xi = \{ p \in \mathbb{P} : p \Vdash _\mathbb{P} f(\check{\xi}) = \check{\alpha} \text{ for some } \alpha \in \kappa \} \).

Claim

\( D_\xi \) is dense.

Proof.
Given \( p \in \mathbb{P} \), let \( G \) be \( \mathbb{P} \)–generic, \( p \in G \). Since \( p \Vdash _\mathbb{P} \dot{f} : \check{\lambda} \to \check{\kappa} \) is a function, \( f_G : \lambda \to \kappa \) is a function. Let \( \alpha < \kappa \) be such that \( f_G(\xi) = \alpha \). By the forcing theorem there is some \( p' \in G \) such that \( p' \Vdash _\mathbb{P} \dot{f}(\check{\xi}) = \check{\alpha} \). Since \( p, p' \in G \), we may find \( q \in \mathbb{P} \), \( q \leq p, p' \). But then \( q \Vdash _\mathbb{P} \dot{f}(\check{\xi}) = \check{\alpha} \).
Given $\xi < \lambda$, let $A_\xi \subseteq D_\xi$ be a maximal antichain in $\mathbb{P}$. By our hypothesis, $|A_\xi| \leq \aleph_0$. Given $\xi < \lambda$, let

$A_\xi = \{ \alpha < \kappa : p \vdash_{\mathbb{P}} \dot{p}(\xi) = \alpha \text{ for some } p \in D_\xi \}$.

Claim

$|A_\xi| \leq \aleph_0$.

Proof.

For every $\alpha \in A_\xi$ let $q_\alpha \in A_\xi$ compatible with some condition $p_\alpha \in \mathbb{P}$ forcing $\dot{f}(\check{\xi}) = \check{\alpha}$. Since $p_\alpha$ and $q_\alpha$ are compatible and $q_\alpha \vdash_{\mathbb{P}} \dot{f}(\xi) = \alpha'$ for some $\alpha'$, it must be that $\alpha' = \alpha$. Hence $A_\xi \subseteq \{ \alpha : p \vdash_{\mathbb{P}} \dot{f}(\check{\xi}) = \alpha \text{ for some } p \in A_\xi \}$. But

$|\{ \alpha : p \vdash_{\mathbb{P}} \dot{f}(\check{\xi}) = \alpha \text{ for some } p \in A_\xi \}| \leq \aleph_0$.

Finally, by the forcing theorem, $\text{range}(\dot{f}_G) \subseteq \bigcup_{\xi < \lambda} A_\xi$ for every $\mathbb{P}$–generic $G$ over $M$, $p \in G$. But

$M \models |\bigcup_{\xi < \lambda} A_\xi| \leq |\lambda \times \aleph_0| = \lambda < \kappa$ implies $\bigcup_{\xi < \lambda} A_\xi \neq \kappa$ (it is easy to see that $|\omega \times \lambda| = \lambda$ for every infinite cardinal $\lambda$ [Exercise.]). Hence $\dot{f}_G$ cannot be onto $\kappa$ for any such $G$. \qed
Corollary

Suppose $M$ is a transitive model of enough of ZFC, $\mathbb{P} \in M$ and $M \models \mathbb{P}$ is c.c.c. Suppose $G$ is $\mathbb{P}$–generic over $M$. Then, for every $\kappa \in M$ such that $M \models \kappa$ is a cardinal, $M[G] \models \kappa$ is a cardinal. In particular, for every ordinal $\alpha \in M$, $\mathcal{N}_\alpha^M = \mathcal{N}_\alpha^{M[G]}$. 
Lemma

(ZFC) For every ordinal $\alpha$, $\text{Add}(\omega, \alpha)$ has the countable chain condition.

Proof.

Suppose, towards a contradiction, that $(p_\nu)_{\nu < \omega_1}$ enumerates an uncountable antichain of $\text{Add}(\omega, \alpha)$. Since $\{\text{dom}(p_\nu) : \nu < \omega_1\}$ is a collection of finite sets, by the $\Delta$–system lemma we know that there is $X \subseteq \omega_1$, $|X| = \aleph_1$, such that $\{\text{dom}(p_\nu) : \nu \in X\}$ forms a $\Delta$–system with root $R$; in other words, for all distinct $\nu, \nu' \in X$, $\text{dom}(p_\nu) \cap \text{dom}(p_\nu') = R$. Note that there are only finitely many functions from the finite set $R$ to $2$ (in fact there are exactly $2^{|R|}$ such functions).

Since $|X| = \aleph_1$ and $2^{|R|}$ is finite, it follows that there are distinct $\nu, \nu' \in X$ such that $p_\nu \upharpoonright R = p_{\nu'} \upharpoonright R$. But then, $p_\nu \cup p_{\nu'} : \text{dom}(p_\nu) \cup \text{dom}(p_{\nu'}) \rightarrow 2$ is a function. Since $q := p_\nu \cup p_{\nu'}$ extends both $p_\nu$ and $p_{\nu'}$, we have that $q \leq \text{Add}(\omega, \alpha) p_\nu, p_{\nu'}$. A contradiction.
Corollary

Suppose $M$ is a transitive model of enough of ZFC, $\kappa \in M$ is such that $M \models \text{“} \kappa \text{ is a cardinal} \text{”}$, and $G$ is an $\text{Add}(\omega, \kappa)$–generic filter over $M$. Then

1. for every ordinal $\lambda \in M$, $M \models \text{“} \lambda \text{ is a cardinal} \text{”}$ if and only if $M[G] \models \text{“} \lambda \text{ is a cardinal} \text{”}$, and
2. $M[G] \models 2^{\aleph_0} \geq \kappa$

In particular, if $\alpha \in \text{Ord} \cap M$ and $\kappa = \aleph_\alpha^M$, then $M[G] \models 2^{\aleph_0} \geq \aleph_\alpha$. 
\[\sigma \text{–closedness and not adding new reals}\]

We see a companion result next: The consistency of ZFC + CH, relative to the consistency of ZFC, using forcing (as we saw, the original proof of this result, due to Gödel, made use of the constructible universe):

**Definition**

A partial order \(\mathbb{P}\) is \(\sigma\text{–closed}\) iff for every \(\leq_{\mathbb{P}}\)–decreasing sequence \((p_n)_{n<\omega}\) of \(\mathbb{P}\)–conditions there is some \(q \in \mathbb{P}\) such that \(q \leq_{\mathbb{P}} p_n\) for all \(n\).
Lemma
Suppose $M$ is a transitive model of enough of ZFC, $\mathbb{P} \in M$ is a forcing notion, $M \models \mathbb{P}$ is $\sigma$–closed, $G$ is $\mathbb{P}$–generic over $M$, and $f : \omega \to \text{Ord}$, $f \in M[G]$. Then $f \in M$.

Proof.
For every $n < \omega$, $D_n = \{ p \in \mathbb{P} : p \Vdash \dot{f}(n) = \alpha \text{ for some } \alpha \} \in M$ is dense. Also, of course $(D_n)_{n<\omega} \in M$. It suffices to see that there is an ordinal $\theta \in M$ such that

$D = \{ p \in \mathbb{P} : p \Vdash \dot{f} = \dot{g} \text{ for some function } g \in V^M_{\theta} \} \in M$ is dense.

For this let $p \in \mathbb{P}$. Build, in $M$, a $\leq_{\mathbb{P}}$–decreasing sequence $(p_n)_{n<\omega}$ of conditions in $\mathbb{P}$ extending $p$ and such that $p_{n+1} \in D_n$ for all $n$.

Let $p_0 = p$. Given $p_n$, we can find $p_{n+1}$ since $D_n$ is dense.

Finally, let $q \leq_{\mathbb{P}} p_n$ for all $n$. $q$ exists since $\mathbb{P}$ is $\sigma$–closed in $M$. But then $q \in D$. 
Definition
Let \( \text{Coll}(\mathcal{P}(\omega), \omega_1) \) be the following partial order:
\[ p \in \text{Coll}(\mathcal{P}(\omega), \omega_1) \text{ iff } p \subseteq \omega_1 \times \mathcal{P}(\omega) \text{ is a function such that } |p| \leq \aleph_0. \]
Given \( \text{Coll}(\mathcal{P}(\omega), \omega_1) \)–conditions \( p_0, p_1, p_1 \leq \text{Coll}(\mathcal{P}(\omega), \omega_1) \) \( p_0 \) iff \( p_0 \subseteq p_1. \)

Lemma
\( \text{Coll}(\mathcal{P}(\omega), \omega_1) \) is \( \sigma \)–closed.

Proof.
Obvious, since the union of a decreasing \( \omega \)–sequence in \( \text{Coll}(\mathcal{P}(\omega), \omega_1) \) is a countable function. \( \square \)
Lemma
For every transitive model $M$ of enough of ZFC and every $r \subseteq \omega$, $r \in M$, $D_r = \{ p \in \text{Coll}(\mathcal{P}(\omega), \omega_1)^M : r \in \text{range}(p) \} \in M$ is a dense subset of $\text{Coll}(\mathcal{P}(\omega), \omega_1)^M$.

Proof.
If $p \in \text{Coll}(\mathcal{P}(\omega), \omega_1)^M$ and $r \notin \text{range}(p)$, then $p \cup \{(\alpha, r)\} \in D$, where $\alpha$ is any ordinal in $\omega_1^M \setminus \text{dom}(p) \neq \emptyset$.

Similarly we can prove:

Lemma
For every transitive model $M$ of enough of ZFC and every $\alpha \in \omega_1^M$, $E_\alpha = \{ p \in \text{Coll}(\mathcal{P}(\omega), \omega_1)^M : \alpha \in \text{dom}(p) \} \in M$ is a dense subset of $\text{Coll}(\mathcal{P}(\omega), \omega_1)^M$. 
Corollary

Let $M$ be a transitive model of enough of ZFC and let $G$ be a generic filter for $\text{Coll}(\mathcal{P}(\omega), \omega_1)^M$. Then $M[G] \models 2^{\aleph_0} = \aleph_1$.

Proof.

By the last two lemmas above, $\bigcup G : \omega_1^M \rightarrow \mathcal{P}(\omega)^M$ is a surjection. Also, by the previous two lemmas, $\mathcal{P}(\omega)^{M[G]} = \mathcal{P}(\omega)^M$ since $\text{Coll}(\mathcal{P}(\omega), \omega_1^M)$ is $\sigma$–closed in $M$ and therefore does not add new functions $f : \omega \rightarrow 2$. But then $M[G]$ thinks that $\bigcup G : \omega_1 \rightarrow \mathcal{P}(\omega)$ is onto all of $\mathcal{P}(\omega)$. □
By the above corollaries, together with the metamathematical considerations at the beginning of this section on forcing, we have proved

- $\text{Con}(\text{ZFC})$ implies $\text{Con}(\text{ZFC} + \text{CH})$.
- $\text{Con}(\text{ZFC})$ implies $\text{Con}(\text{ZFC} + 2^{\aleph_0} \geq \aleph_2)$, $\text{Con}(\text{ZFC} + 2^{\aleph_0} \geq \aleph_3)$, $\text{Con}(\text{ZFC} + 2^{\aleph_0} \geq \aleph_{2454536254465})$, $\text{Con}(\text{ZFC} + 2^{\aleph_0} \geq \aleph_{\aleph_785665})$, etc.
Remark: It is not difficult to prove, by the same forcing construction we have seen (adding many Cohen reals) using a slightly more refined analysis, that $2^{\aleph_0}$ can consistently be exactly $\aleph_2$, $\aleph_3$, $\aleph_{2454536254465}$, $\aleph_{\aleph_{785665}}$, etc.

It is also easy to change the value of $2^{\aleph_1}$, $2^{\aleph_2}$, $2^{\aleph_{\omega+1}}$, and so on, by forcing. And, in fact, any constellation of values is possible for cardinals as the above (and many others), subject to some very simple rules. For example, it is consistent – modulo Con(ZFC) of course – that

$2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_27 + 2^{\aleph_2} = 2^{\aleph_3} = \aleph_28 + 2^{\aleph_{\omega+1}} = \aleph_{\omega_1}$, etc.
On the other hand, there are very deep and surprising cardinal arithmetical facts one can prove in ZFC for other cardinals! For example:

**Theorem**

*(Shelah) (ZFC)* Suppose $2^{\aleph_n} < \aleph_\omega$ for all $n < \omega$. Then $2^{\aleph_\omega} < \aleph_\omega_4$.

A big open question in “pcf theory” is whether the above bound $\aleph_\omega_4$ can be improved to something like $\aleph_\omega_1$. 
Forcing has applications not only in cardinal arithmetical. In fact it can be used to build models of set theory with many different properties.

One example (J. Moore): If $M$ is a transitive countable model of (a fragment of) $\text{ZFC} + \text{LC}$, where $\text{LC}$ is some (relatively weak) “large cardinal axiom”, then there is a forcing extension $M[G]$ such that $M[G]$ thinks that there is a basis of exactly 5 elements for the uncountable linear orders; in other orders, in $M[G]$ it holds that there is $B = \{L_0, L_1, L_2, L_3, L_4\}$ such that each $L_i$ is an uncountable linear order, and such that if $L$ is any uncountable linear order, then $L$ contains an isomorphic copy of at least one $L_i$. In this model, $2^{\aleph_0} > \aleph_1$. In fact, $2^{\aleph_0} = \aleph_1$ implies that if $B$ is a basis for the uncountable linear orders, then $|B| = 2^{\aleph_1}$. 
We have seen that ZFC does not decide the value of \(2^{\aleph_0}\) (i.e., does not decide the ordinal \(\alpha\) such that \(2^{\aleph_0} = \aleph_\alpha\)). Also, ZFC does not decide whether or not there is a 5-element basis for the uncountable linear orders, as well as infinitely many other questions in mathematics (hundreds or thousands of these questions have been explicitly considered by mathematicians).

Given the independence phenomenon, should we gather these questions as meaningless?
Not necessarily. We can hope to find *natural* axioms which, when added to ZFC, decide these questions. The most successful and so far best understood family of such axioms come under the name of “large cardinal axioms”. The weakest such axiom asserts that there is an uncountable cardinal $\kappa$ such that

- $\kappa$ is a limit cardinal, and
- $\kappa$ is *regular* (i.e., $\kappa$ is not the union of less than $\kappa$–many sets of size less than $\kappa$).

Such a $\kappa$ is called a *(strongly) inaccessible cardinal*. 
It is not possible to prove
\[ \text{Con}(\text{ZFC}) \iff \text{Con}(\text{ZFC} \ + \ \text{"There is an inaccessible cardinal"}) \]
because \( \text{ZFC} + \text{"There is an inaccessible cardinal"} \) proves \( \text{Con}(\text{ZFC}) \) (unless we can prove \( \neg \text{Con}(\text{ZFC}) \), that is): The reason is that if \( \kappa \) is inaccessible, then \( V_\kappa \models \text{ZFC} \). Now suppose towards a contradiction that we can prove the arithmetical statement
\[ \text{Con}(\text{ZFC}) \iff \text{Con}(\text{ZFC} + \text{"There is an inaccessible cardinal"}) \]
in, say, Peano Arithmetic or \( \text{ZFC} \). But
\[ \text{ZFC} + \text{"There is an inaccessible cardinal"} \]
proves \( \text{Con}(\text{ZFC}) \). Hence
\[ \text{ZFC} + \text{"There is an inaccessible cardinal"} \]
proves \( \text{Con}(\text{ZFC} + \text{"There is an inaccessible cardinal"}) \), so
\[ \text{ZFC} + \text{"There is an inaccessible cardinal"} \]
is inconsistent by Gödel’s 2nd Incompleteness Theorem. Hence \( \text{ZFC} \) would be inconsistent.
We say that $\text{ZFC} + \text{“There is an inaccessible cardinal”}$ is strictly stronger than $\text{ZFC}$.

Similarly:

- $\text{ZFC} + \text{“There is a weakly compact cardinal”}$ is strictly stronger than $\text{ZFC} + \text{“There is an inaccessible cardinal”}$.
- $\text{ZFC} + \text{“There is a measurable cardinal”}$ is strictly stronger than $\text{ZFC} + \text{“There is a weakly compact cardinal”}$.
- $\text{ZFC} + \text{“There is a Woodin cardinal”}$ is strictly stronger than $\text{ZFC} + \text{“There is a measurable cardinal”}$.
- $\text{ZFC} + \text{“There is a supercompact cardinal”}$ is strictly stronger than $\text{ZFC} + \text{“There is a Woodin cardinal”}$.
- $\text{ZFC} + \text{“There is a huge cardinal”}$ is strictly stronger than $\text{ZFC} + \text{“There is a supercompact cardinal”}$.

... 

The theories of the form $\text{ZF} + \text{LC}$ (or $\text{ZFC}+\text{LC}$), where LC is a large cardinal axiom, form a hierarchy of theories naturally linearly ordered by consistency strength.
We say that ZFC + “There is an inaccessible cardinal” is strictly stronger than ZFC.
Similarly:

- ZFC + “There is a weakly compact cardinal” is strictly stronger than ZFC + “There is an inaccessible cardinal”.
- ZFC + “There is a measurable cardinal” is strictly stronger than ZFC + “There is a weakly compact cardinal”.
- ZFC + “There is a Woodin cardinal” is strictly stronger than ZFC + “There is a measurable cardinal”.
- ZFC + “There is a supercompact cardinal” is strictly stronger than ZFC + “There is a Woodin cardinal”.
- ZFC + “There is a huge cardinal” is strictly stronger than ZFC + “There is a supercompact cardinal”.
- ...

The theories of the form ZF + LC (or ZFC+LC), where LC is a large cardinal axiom, form a hierarchy of theories natural linearly ordered by consistency strength.
We say that \( \text{ZFC} + \text{“There is an inaccessible cardinal”} \) is strictly stronger than \( \text{ZFC} \).

Similarly:

- \( \text{ZFC} + \text{“There is a weakly compact cardinal”} \) is strictly stronger than \( \text{ZFC} + \text{“There is an inaccessible cardinal”} \).
- \( \text{ZFC} + \text{“There is a measurable cardinal”} \) is strictly stronger than \( \text{ZFC} + \text{“There is a weakly compact cardinal”} \).
- \( \text{ZFC} + \text{“There is a Woodin cardinal”} \) is strictly stronger than \( \text{ZFC} + \text{“There is a measurable cardinal”} \).
- \( \text{ZFC} + \text{“There is a supercompact cardinal”} \) is strictly stronger than \( \text{ZFC} + \text{“There is a Woodin cardinal”} \).
- \( \text{ZFC} + \text{“There is a huge cardinal”} \) is strictly stronger than \( \text{ZFC} + \text{“There is a supercompact cardinal”} \).
- ... 

The theories of the form \( \text{ZF} + \text{LC} \) (or \( \text{ZFC} + \text{LC} \)), where LC is a large cardinal axiom, form a hierarchy of theories natural linearly ordered by consistency strength.
It is a remarkable empirical fact that all ‘natural’ theories arising in mathematics can be interpreted relative to some such theory, and that in many cases they can be proved to be equiconsistent with such a theory. In this sense, large cardinal axioms provide a very natural template for extending ZF or ZFC.
Sufficiently strong large cardinal axioms have also remarkable “astrological” properties: They prove things about ‘simply definable’ sets of reals that ZFC does not decide, if consistent.

For example: If there are infinitely many Woodin cardinals, then every projective set of reals has the usual regularity properties (it is Lebesgue measurable, has the Baire property, and has the perfect set property).

On the other hand, in the absence of such large cardinals it is possible that there be a closed set $X \subseteq \mathbb{R}^3$ such that $Y$ is not Lebesgue measurable, where $Y$ is the projection to $\mathbb{R}$ of the complement, in $\mathbb{R}^2$, of the projection of $X$ to $\mathbb{R}^2$. This $Y$ is, by construction, a projective set, in fact quite low in the projective hierarchy.
(Un)fortunately one can prove that large cardinal axioms do not decide $2^{\aleph_0}$. In fact, all natural large cardinals $\kappa$ are immune to ‘small’ forcing, i.e., forcing of size less than $\kappa$ (precisely the kind of forcing we have used to tinker with the value of $2^{\aleph_0}$).

For this one has to look at other families of axioms, compatible with large cardinal axioms, for example “forcing axioms”. But this is another story.