Large cardinals, forcing axioms, and mathematical realisms

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Incompleteness

A sociological fact: Zermelo-Fraenkel set theory with the Axiom of Choice, \textit{ZFC}, is the standard foundation for mathematics.

As initially proved by Kurt Gödel in the 1930’s with his First Incompleteness Theorem, \textit{ZFC} is intrinsically incomplete, in the sense that for every reasonable consistent extension \( T \) of \textit{ZFC} there are statements in the language of \( T \) that \( T \) does nor decide.

In 1963, Paul Cohen invented the forcing method and showed, with the use of this method, the independence of Cantor’s Continuum Hypothesis (\textit{CH}) from \textit{ZFC}. Many other (natural mathematical) statements have been shown to be undecided by \textit{ZFC}. 
It is therefore natural to search for additional natural axioms which, when added to ZFC, suffice to decide such questions. The need to find such axioms is all the more urgent if we assume a realist standpoint, whereby the cumulative hierarchy of sets describes a uniquely specifiable object. According to such a view, a question such as “How many real numbers are there?”, which of course CH answers, should have a unique solution in this hierarchy.
Large cardinal axioms

Large cardinal axioms form a natural hierarchy of axioms extending ZFC. They indeed tend to build a hierarchy, in the sense that any two of these axioms, $A$ and $A'$, are compatible, and in fact often comparable (i.e., $A$ implies $A'$ or $A'$ implies $A$).

These axioms lie beyond the scope of what ZFC can prove. In fact they transcend ZFC in that they cannot be proved to be consistent assuming just the consistency of ZFC (in this sense, they are different from axioms such as CH, or its negation, both of which can be proved, by forcing, to be consistent together with ZFC). This is essentially the content of Gödel’s Second Incompleteness Theorem.
Moreover, it is a remarkable empirical fact that all natural mathematical theories can be interpreted within $\text{ZFC}+A$ for some suitable large cardinal axiom $A$.

Unfortunately, despite their realist appeal conferred to them by the above (especially their lying in a natural hierarchy), large cardinal axioms do not settle such statements as $\text{CH}$:

Suppose $\kappa$ is a large cardinal. You can force $\text{CH}$ (or $\neg \text{CH}$) by very small forcing, of size less than $\kappa$. But then $\kappa$ retains its large cardinal property in the extension.
Enforced realism

Let us pretend we are realists not about the set-theoretic universe (CH undecided by ZFC), but are realists about arithmetic ($2 + 2$ really is 4). According to this view, Goldbach's conjecture should be definitely true or false, etc. Believe in the complete existence of arithmetic enforces us, though, to more realism:

Take an arithmetical statement like Con(ZFC) or Con(ZFC + There is a supercompact cardinal). Most people would agree that Con(ZFC) is true. And probably many set-theorists would agree that Con(ZFC + There is a supercompact cardinal) is true. But what reason do we have to believe in the truth of these arithmetical if not the belief in the existence of a realm in which these theories are true?
Large cardinal axioms: A template

Most large cardinal axioms can be phrased in the form “There is an elementary embedding $j : M \rightarrow N$, $j \neq id$, where $M$ and $N$ are such that ....”.

The strongest axioms are those where $M$ is the entire universe ($V$). And the closer $N$ is to $V$, the stronger the axiom.

Examples:

- $\kappa$ is a measurable cardinal iff there is an elementary embedding $j : V \rightarrow M$, for a transitive class $M$, such that $\text{crit}(j) = \kappa$, where $\text{crit}(j)$ is the first ordinal $\alpha$ such that $\alpha < j(\alpha)$.

- $\kappa$ is a supercompact cardinal iff for every $\lambda > \kappa$ there is an elementary embedding $j : V \rightarrow M$, for a transitive class $M$, such that $\text{crit}(j) = \kappa$ and $M$ is closed under $\lambda$–sequences.
(Caveat: These are second order assertions, and hence not part of the language of set theory. However, there are ways to may first order sense of such formulations.)

A natural limit of this family of large cardinal axioms:

“There is a Reinhardt cardinal.”

\( \kappa \) is a *Reinhardt cardinal* if \( \kappa = \text{crit}(j) \) for some elementary embedding \( j : V \rightarrow V, j \neq \text{id} \).

Defined by Reinhardt in about 1967. However ...

**Theorem**

(Kunen, 1971) (ZFC) *There are no Reinhardt cardinals*
(Caveat: These are second order assertions, and hence not part of the language of set theory. However, there are ways to may first order sense of such formulations.)

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In fact, Kunen’s proof shows the following (stronger statement):

There is no elementary embedding \( j : V_{\lambda+2} \rightarrow V_{\lambda+2} \), for any ordinal \( \lambda \), \( j \neq id \).

On the other hand, despite a lot of work, no inconsistency has been found in ZFC+ “There is some \( \lambda \) for which there is elementary embedding \( j : V_{\lambda+1} \rightarrow V_{\lambda+1}, j \neq id \).”

And even from stronger theories.
An epistemological blind spot

We know that $\text{Con}(\text{ZFC} + \text{There is a Reinhardt cardinal})$ is a false arithmetical statement. Where is the border between consistency and inconsistency? Is $\text{Con}(\text{ZFC} + \text{There is a supercompact cardinal})$ on the inconsistent side? What about $\text{Con}(\text{ZFC})$? $\text{Con}(\text{Peano Arithmetic})$ surely is consistent (after all we agreed that arithmetic is real)! This border must be well–defined if arithmetic is real. But by Gödel’s Second Incompleteness Theorem we will never have mathematical access to it.
Interestingly, the following question remains open.

**Question**

*Does ZF prove the nonexistence of Reinhardt cardinals?*

This question motivates studying large cardinals beyond Choice.
A word or two on applications of large cardinals and / or forcing

ZFC, if consistent, doesn’t settle any of the following questions:

(1) Is $\aleph_1$ the cardinality of $\mathbb{R}$? Is it $\aleph_{24567}$?

(2) Is the real line the only (up to order–isomorphism) complete linear order without end–points and without an uncountable collection of pairwise disjoint nonempty intervals?

(3) Is there a list of 5 uncountable linear orders such that every uncountable linear order contains a suborder order–isomorphic to some of the members of the list?
(4) Let $X$ be a closed subset of $\mathbb{R}^3$. Let $Y$ be the projection of $X$ on $\mathbb{R}^2$. Let $Z$ be the projection of $\mathbb{R}^2 \setminus Y$ on $\mathbb{R}$. Is $Z$ necessarily Lebesgue measurable?

(5) Let $X$ be a set. $\mu : \mathcal{P}(X) \longrightarrow \mathbb{R}$ is a (probabilistic) $\sigma$–additive measure on $X$ iff:

(a) $\mu(\emptyset) = 0$ and $\mu(X) = 1$.
(b) If $\mu(\{a\}) = 0$ for all $a \in X$.
(c) If $(Y_n)_{n<\omega}$ is a sequence of pairwise disjoint subsets of $X$, then $\mu(\bigcup_n Y_n) = \sum_n \mu(Y_n)$.

Lebesgue measure (restricted to subsets of the interval $[0, 1]$) satisfies (a)–(c) but not every subset of $[0, 1]$ is Lebesgue measurable.

Is there any $\sigma$–additive measure on $[0, 1]$?
It turns out that:

(a) The theories
   • $\text{ZF}$,
   • $\text{ZFC} + \text{“The answer to question } (n) \text{ is YES”, for } n = 1, 2,$
     and
   • $\text{ZFC} + \text{“The answer to question } (n) \text{ is NO” for } n = 1, 2, 3, 4, 5$

   are equiconsistent.

(b) $\text{Con}(\text{ZFC} + \text{“There is a weakly compact cardinal”})$ is more than enough to prove
    $\text{Con}(\text{ZFC} + \text{“The answer to question (3) is YES”})$, and it is not known if, for example, $\text{Con}(\text{ZFC})$ suffices.
(c) The theories
   • ZFC + “There is an inaccessible cardinal”, and
   • ZFC + “The answer to Question (4) is YES”

are equiconsistent.

(d) The theories
   • ZFC + “There is a measurable cardinal” and
   • ZFC + “The answer to Question (5) is YES”

are equiconsistent.
Astrological properties of large cardinals

A lot of mathematics takes place in rather small initial segments of the cumulative hierarchy. $V_{\omega+n}$, for some small $n < \omega$, is where the situations described in all of the questions (1)–(5) live, and where most of ordinary analysis, etc., live. It is therefore a remarkable fact that the mere existence of large certain large cardinals, very high up, has a direct influence on the properties of ‘down to earth’ things like sets of reals.

Example:

**Theorem**

(ZFC) Suppose there are infinitely many Woodin cardinals. Then every projective set of reals (i.e., every set obtained from a closed subset of $\mathbb{R}^n$, for some $n$, in finitely many steps by projecting and taking complements) is Lebesgue measurable.
In fact this theorem is a corollary of the following.

**Theorem**
*(ZFC) Woodin, Martin–Steel, mid 1980’s. The following are equivalent.*

1. Projective Determinacy.
2. For every $n$ and every real $x$ there is an iterable inner model with $n$ Woodin cardinals and containing $x$. 
Available methods in set theory

Our two main methods known to prove consistency (or even independence) in set theory:

(1) Large cardinal axioms: They even imply arithmetical statements (\(\text{Con}(\text{ZFC}),\ldots\)). More interestingly, they have implications at the level of definable sets of reals (all projective sets of reals are Lebesgue measurable, \(\ldots\)). On the other hand, they do not decide things like \(\text{CH}\).

(2) Forcing: As mentioned, forcing enables us to build models of mathematical statements like \(\text{CH}, \neg \text{CH}\), and many others. On the other hand, forcing extensions leave arithmetic unchanged. And they preserve the ambient large cardinals (at least if there is a proper class of relevant large cardinals). Hence, forcing extensions preserve everything these large cardinals imply.
Axiom discrimination via set–forcing

To show that ZFC does not imply some statement \( \sigma \), build a suitable forcing notion \( P \) in the universe, \( V \), so that if \( G \) is a \( P \)–generic filter over \( V \),

\[
V[G] \models \neg \sigma
\]

The point is of course that

\[
V \models \text{ZFC} \iff V[G] \models \text{ZFC}
\]

Classical example: The Continuum Hypothesis \( 2^{\aleph_0} = \aleph_1 \) (CH) is independent from ZFC, i.e., \( \text{ZFC} \nvdash \text{CH} \) and \( \text{ZFC} \nvdash \neg \text{CH} \).
We can replace in the above ZFC with, say,

\[ \text{ZFC + “There is proper class of cardinals } \kappa \text{ such that ...”}, \]

where ... is some sensible large cardinal property (e.g. ‘\( \kappa \) is measurable’, ‘\( \kappa \) is supercompact’, etc.). The point is, again, that any forcing notion \( P \in V \) will preserve the relevant large cardinal property of all \( \kappa \) larger than \( P \).

In the above, the axioms of ZFC, or “There is proper class of cardinals \( \kappa \) such that ...”, are naturally perceived as being truer or more natural than, say, CH or \( \neg \text{CH} \): All of those axioms are immune to forcing, whereas CH and \( \neg \text{CH} \) are not.

Famously, large families of statements have been shown to enjoy, in the presence of large cardinals, the same immunity to forcing than ZFC or your favourite large cardinal axiom:
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Chang models

Given a cardinal $\lambda$, the $\lambda$–Chang model is

$$C_\lambda = L([\text{Ord}]^{\leq \lambda})$$

= the $\subseteq$–minimal inner model of ZF closed under $\leq \lambda$–sequences.

$C_\lambda$ need not satisfy the Axiom of Choice; for instance, if there are $\lambda^+$–many measurable cardinals, then $C_\lambda \models \neg \text{AC}$ (Kunen).

Classical analysis (in $V$) lives in $C_\omega$; in fact, already $L(\mathbb{R})$ contains all reals and all reasonably definable (=$\text{projective}$) sets of reals, and much more, and $L(\mathbb{R}) \subseteq C_\omega$ is definable in $C_\omega$ ($L(\mathbb{R})$ is the $\subseteq$–minimal inner model of ZF containing all reals).
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($L(\mathbb{R})$ is the $\subseteq$–minimal inner model of ZF containing all reals).
Large cardinals give us completeness, relative to our best method (forcing), so long as we restrict to statements expressible in the $\omega$–Chang model. Specifically:

**Theorem**

(Woodin, around 1985) (ZFC) Suppose there is a supercompact cardinal or a proper class of Woodin cardinals. Then for every set–forcing $\mathcal{P}$ and every $\mathcal{P}$–generic $G$ over $V$,

\[
(c^V_\omega; \in, r)_{r \in \mathbb{R}^V} \equiv (c^{V[G]}_\omega; \in, r)_{r \in \mathbb{R}^V}
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In particular, all propositions of classical analysis are immune to set–forcing.
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Most of the original mathematical work I will mention in this talk deals with the problem of extending Woodin’s absoluteness result.
Definition (Woodin) (ZF) Given an ordinal $\alpha$, a cardinal $\kappa$ is $V_\alpha$–supercompact iff there is some $\beta$ and some elementary embedding $j : V_\beta \rightarrow M$ with critical point $\kappa$ and such that $V_\alpha M \subseteq M$ and $j(\kappa) > \alpha$.

$\kappa$ is supercompact iff it is $V_\alpha$–supercompact for all $\alpha$.

If AC holds, this definition coincides with the usual definition.
Theorem
(ZF + DC)

(1) Suppose there is, for every $\Pi_2$ definable closed and unbounded class of ordinals $C$, a $V_{\kappa+2}$–supercompact cardinal $\kappa$ such that $\kappa \in C$. Suppose $\mathcal{P}$ is a set–forcing preserving DC. Then, for every $\mathcal{P}$–generic $G$ over $V$,

$$(C^V_\omega; \in, r)_{r \in R^V} \equiv (C^V[G]_\omega; \in, r)_{r \in R^V}$$

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Theorem
\[(ZF + DC) \text{ Suppose there is a supercompact cardinal } \kappa.\]
Suppose \( \mathcal{P} \) is a set–forcing preserving DC. Then, for every \( \mathcal{P} \)–generic \( G \) over \( V \),

- \((L(\mathbb{R})^V; \in, r)_{r \in \mathbb{R}^V} \equiv (L(\mathbb{R})^{V[G]}; \in, r)_{r \in \mathbb{R}^V}, \text{ and}\)
- the Axiom of Determinacy holds in \( L(\mathbb{R})^{V[G]} \).
Most of analysis (including descriptive set theory) can be naturally carried out in $\text{ZF} + \text{DC}$.

This makes it natural to enquire whether generic absoluteness for the theory of an inner model containing analysis can be derived over the base theory $\text{ZF} + \text{DC}$ (Is full $\text{AC}$ an overkill, as of course is $\text{CH}$, or $\neg \text{CH}$, etc?)
Limitations

$\Sigma_2$–generic absoluteness for the $\omega$–Chang model may fail for set–forcing destroying DC:

Suppose $V \models ZFC$. Let $M$ be the symmetric submodel of a generic extension of $V$ by the Levy collapse $\mathcal{P} = \text{Coll}(\omega_1, <\aleph_{\omega_1})$ given by the filter of subgroups of permutations of $\mathcal{P}$ generated by $G_\alpha = \{ \pi \in \text{Aut}(\mathcal{P}) : \pi(p) = p \text{ for all } p \in \text{Coll}(\omega_1, \alpha) \}$ (for $\alpha < \aleph_{\omega_1}$).

As in the classical Feferman–Levy model for $\omega_1$ singular, $\aleph_{\omega_1}^V = \omega_2^M$ and hence $M$ thinks that $\omega_2$ has cofinality $\omega_1$. 
$M$ satisfies $\text{DC}$ since $\mathcal{P}$ is $\sigma$–closed in $V$.

All supercompact cardinals $\kappa$ in $V$ remains supercompact in $M$, essentially since $|\mathcal{P}| < \kappa$ in $V$.

But then generic absoluteness for the $\Sigma_2$–theory of the Chang model has to fail between $M$ and the extension of $M$ by $\text{Coll}(\omega, \omega_1)$, the collapse of $\omega_1^M$ ($= \omega_1^V$) to $\omega$ with finite conditions, since $\mathcal{C}_\omega^M$ thinks that $\omega_1$ is regular whereas $\mathcal{C}_\omega^{\text{Coll}(\omega, \omega_1)}$ thinks that $\omega_1$ ($= \omega_2^M$) is singular.
Question

Can one get these generic absoluteness results assuming more modest large cardinal hypotheses (in the region of Woodin cardinals)?
II: Back in ZFC. Larger inner models

[This second part is joint work with Matteo Viale.]
Forcing axioms

Given a forcing notion $\mathbb{P}$ and a cardinal $\lambda$, $\text{FA}_\lambda(\mathbb{P})$ means: For every collection $\{D_i : i < \lambda\}$ of dense subsets of $\mathbb{P}$ there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_i \neq \emptyset$ for all $i < \lambda$.

Given a class $\Gamma$ of forcing notions, $\text{FA}_\lambda(\Gamma)$ means $\text{FA}_\lambda(\mathbb{P})$ for all $\mathbb{P} \in \Gamma$.

Forcing axioms are to be seen as maximality / saturation principles relative to forcing; for instance, the version of $\text{FA}_\lambda(\mathbb{B})$ where one allows only dense sets generated by sets $X \subseteq \mathbb{P}$ of size at most $\lambda$ is equivalent to: If $G$ is $\mathbb{P}$–generic over $V$, then

$$H(\lambda^+)^V \equiv_{\Sigma_1} H(\lambda^+)^{V[G]}$$

(Bagaria). Note this is a weak local form of generic absoluteness for the $\lambda^+$–Chang model (since $H(\lambda^+) \subseteq C_{\lambda^+}$ is definable in $C_{\lambda^+}$).
FA(\(P\))_0 is true in ZFC for every forcing notion \(P\), and
FA_{\aleph_1}(\text{Coll}(\omega, \omega_1)) always fails, for example,
where \text{Coll}(\omega, \omega_1) is the collapse of \(\omega_1\) to \(\omega\) with finite conditions.

Set–theorists have isolated various classes \(\Gamma\) for which FA_{\aleph_1}(\Gamma) is con-
sistent, starting with FA_{\aleph_1}(\{P : P \text{ c.c.c.}\}) = Martin’s Axiom at \(\omega_1\).

There is a ZFC–provably \(\subseteq\)–maximal class \(\Gamma_{\text{max}}\) such that
FA_{\aleph_1}(\Gamma_{\text{max}}) is consistent. \(\Gamma_{\text{max}}\) is

\[ \text{SSP} = \{ P : P \text{ preserves all stationary subsets of } \omega_1 \} \]

FA_{\aleph_1}(\text{SSP}) was isolated and proved consistent, relative to a
FA_{\aleph_1}(\text{SSP}) is known as Martin’s Maximum (MM).
Martin’s Maximum++ (MM++)

MM++: For every $\mathbb{P} \in \text{SSP}$, every collection $\{D_i : i < \omega_1\}$ of dense subsets of $\mathbb{P}$ and every collection $\{\dot{S}_i : i < \omega_1\}$ of $\mathbb{P}$-names for stationary subsets of $\omega_1$ there is a filter $G \subseteq \mathbb{P}$ such that

- $G \cap D_i \neq \emptyset$ for all $i$, and
- $\{\nu < \omega_1 : (\exists p \in G)(p \forces \nu \in \dot{S}_i)\}$ is stationary for each $i$.

The usual forcing construction showing the consistency of MM shows in fact the consistency of MM++.
**MM**++ as a density property of **SSP**.

**Proposition**

(ZFC + There is a proper class of Woodin cardinals) The following are equivalent.

1. **MM**++
2. For every \( \mathbb{P} \in \text{SSP} \) there is a pre-saturated \( \omega_2 \)–tower \( \mathcal{T} \in \text{SSP} \) forcing–equivalent to \( \mathbb{P} \ast \dot{\mathbb{Q}} \), where \( \Vdash_{\mathbb{P}} \dot{\mathbb{Q}} \in \text{SSP} \).

The proof is a standard stationary tower argument.
Viale defines a strengthening $\text{MM}^{+++}$ of the above characterisation (2) of $\text{MM}^{++}$ in the presence of large cardinals: He defines the notion of *strongly pre-saturated tower* and

$\text{MM}^{+++}$: For every $\mathbb{P} \in \text{SSP}$ there is a strongly pre-saturated $\omega_2$–tower $\mathcal{T} \in \text{SSP}$ forcing–equivalent to $\mathbb{P} \ast \mathbb{Q}$, where $\models_{\mathbb{P}} \mathbb{Q} \in \text{SSP}$.

He proves that, granting large cardinals, $\text{MM}^{+++}$ can be forced by SSP forcing, and the following generic absoluteness result for the $\omega_1$–Chang model.

**Theorem**

*(Viale) (ZFC + There is a proper class of super–huge cardinals)*

Suppose $\text{MM}^{+++}$ holds. Let $\mathbb{P} \in \text{SSP}$, let $G$ be $\mathbb{P}$–generic over $V$, and suppose $V[G] \models \text{MM}^{+++}$. Then

$$(C_{\omega_1}^V \models \in, r)_{r \in \mathcal{P}(\omega_1)^V} \equiv (C_{\omega_1}^{V[G]} \models \in, r)_{r \in \mathcal{P}(\omega_1)^V}$$
Suitable classes and category forcing axioms

In joint work with Viale, we have extended his $\text{MM}^{+++}$–theory to an abstract setting: We define the notion of $\lambda$–suitable class of forcing notions and, given an arbitrary $\lambda$–suitable class $\Gamma$,

1. define the $\lambda$–Category Forcing Axiom for $\Gamma$, $\text{CFA}_\lambda(\Gamma)$,
2. show its consistency modulo large cardinals, and
3. prove that, in the presence of large cardinals, if $\text{CFA}_\lambda(\Gamma)$ holds, $\mathbb{P} \in \Gamma$ forces $\text{CFA}_\lambda(\Gamma)$, and $G$ is $\mathbb{P}$–generic over $V$, then

\[(C^V_\lambda; \in, r)_{r \in \mathcal{P}(\lambda)^V} \equiv (C^V[G]_\lambda; \in, r)_{r \in \mathcal{P}(\lambda)^V}\]

Modulo large cardinals, $\text{SSP}$ is $\omega_1$–suitable, and $\text{MM}^{+++}$ is $\text{CFA}_{\kappa_1}(\text{SSP})$. 
We have found $\aleph_1$–many $\omega_1$–suitable classes. Some examples:

- $\alpha$–Proper, for a fixed nonzero indecomposable ordinal $\alpha < \omega_1$.
- $\alpha$–Proper $\cap \omega\omega$–bounding, for a fixed nonzero indecomposable ordinal $\alpha < \omega_1$.
- $\alpha$–Proper $\cap$ Suslin tree preserving, for a fixed nonzero indecomposable ordinal $\alpha < \omega_1$.
- $\alpha$–Proper $\cap \omega\omega$–bounding $\cap$ Suslin tree preserving, for a fixed nonzero indecomposable ordinal $\alpha < \omega_1$.
- $\alpha$–Semiproper, for a fixed nonzero indecomposable ordinal $\alpha < \omega_1$.
- ...
- $S$–cond (a class containing Namba forcing, essentially due to Shelah)
Incompatible category forcing axioms
All these category forcing axioms are pairwise incompatible.

Some examples:

- \( \text{MM}^{+++} \implies \text{MM}, \) and \( \text{MM} \implies \delta_2^1 = \omega_2 \) by well–known result of Woodin.

- If there is a super–huge cardinal \( \delta \) and \( \text{CFA}_{\aleph_1}^{\text{Proper}} \) \((= \text{PFA}^{+++})\) holds, then \( \delta_2^1 < \omega_2 \): If \( \delta \) is super–huge, then \( \mathbb{U}_\delta^{\text{Proper}} \) is proper and collapses \( \omega_2^V \) to \( \aleph_1 \). Also, by a result of Neeman-Zapletal, under this large cardinal hypothesis (a proper class of Woodin cardinals suffices), if \( \mathcal{P} \) is a proper poset and \( G \) is \( \mathcal{P} \)–generic over \( V \), then there is an elementary embedding between \( L(\mathbb{R})^V \) and \( L(\mathbb{R})^V[G] \) which is the identity on the ordinals. Hence \( V^{\mathbb{U}_\delta^{\text{Proper}}_\delta} \models \delta_2^1 < \omega_2 \). Since “\( \delta_2^1 < \omega_2 \)” is expressible in \( C_{\omega_1} \), \( V^{\text{Proper}_{\delta}} \models \text{PFA}^{+++} \) and \( V \models \text{PFA}^{+++} \), by the generic absoluteness theorem we that \( V \models \delta_2^1 < \omega_2 \).
There is nothing strange about the incompatibility of these $CFA_\lambda(\Gamma)$’s:

$CFA_\lambda(\Gamma)$ is a statement of the form $(\forall B \in \Gamma)(\exists C \in \Gamma), \ldots$

Hence, even if $\Gamma \subseteq \Gamma'$, this does not obviously entail $CFA_\lambda(\Gamma') \implies CFA_\lambda(\Gamma)$ (in fact it does not).
Question
Are there \( \lambda \)-suitable classes for \( \lambda > \omega_1 \) ?

Question

MM is the provably maximal consistent forcing axiom of the form \( FA_{\aleph_1}(\Gamma) \). Is there any provably maximal consistent forcing axiom of the form \( FA_{\kappa}(\Gamma) \) for \( \kappa > \aleph_1 \)? Perhaps even a unique one?

The most obvious candidate,

\[
FA_{\kappa}(\{ \mathbb{P} : \mathbb{P} \text{ preserves stationary sets of } \mu \text{ for all } \mu \leq \kappa \})
\]

is inconsistent (Shelah).
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is inconsistent (Shelah).
One conclusion of 2nd part:

Completeness modulo forcing is not by itself a good criterion for choosing new (true) axioms if we want a univocal description of the universe. Maximality is better suited for this. Completeness modulo forcing can be a pleasant added value of maximality axioms, though. \( \text{MM}^{+++} \) looks more natural than any other CFA(\( \Gamma \)), since \( \text{MM}^{+++} \) implies the maximal forcing axiom \( \text{MM} \) whereas the other CFA\( \kappa_1(\Gamma) \)'s don’t.

Question

Does \( \text{MM}^{+++} \) imply Woodin axiom (\( \ast \))?
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Completeness modulo forcing is not by itself a good criterion for choosing new (true) axioms if we want a univocal description of the universe. Maximality is better suited for this. Completeness modulo forcing can be a pleasant added value of maximality axioms, though. $\text{MM}^{+++}$ looks more natural than any other $\text{CFA}(\Gamma)$, since $\text{MM}^{+++}$ implies the maximal forcing axiom $\text{MM}$ whereas the other $\text{CFA}_{\aleph_1}(\Gamma)$’s don’t.

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One conclusion of 2nd part:

Completeness modulo forcing is not by itself a good criterion for choosing new (true) axioms if we want a univocal description of the universe. Maximality is better suited for this. Completeness modulo forcing can be a pleasant added value of maximality axioms, though. $\text{MM}^{+++}$ looks more natural than any other CFA($\Gamma$), since $\text{MM}^{+++}$ implies the maximal forcing axiom $\text{MM}$ whereas the other CFA$_{N_1}(\Gamma)$’s don’t.

Question

Does $\text{MM}^{+++}$ imply Woodin axiom (⋆)?
Another maximality principle: Shelah’s dream (?)

Call a $\Sigma_2$ statement $(\exists \beta) V_\beta \models \sigma$ (for a sentence $\sigma$) nice if for every ordinal $\alpha$ there is a set–forcing preserving $V_\alpha$ and forcing $(\exists \beta) V_\beta \models \sigma$.

Question
(Woodin) Let $\#$ be the statement “Every nice $\Sigma_2$–statement is true.” Is $\#$ consistent with ZFC?

Note: $\#$ implies there is some $\kappa$ such that $2^\kappa = \kappa^+$, and some $\kappa$ such that $2^\kappa = \kappa^{+7567}$, and some $\kappa$ such that $\diamondsuit_\kappa$ holds, and some $\kappa$ such that $\diamondsuit_\kappa$ fails, and so on...

“Everything possible is true at some level of the universe.”
Another maximality principle: Shelah’s dream (?)

Call a $\Sigma_2$ statement $(\exists \beta) V_\beta \models \sigma$ (for a sentence $\sigma$) *nice* if for every ordinal $\alpha$ there is a set–forcing preserving $V_\alpha$ and forcing $(\exists \beta) V_\beta \models \sigma$.

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**Question**

*(Woodin)* Let $\mathcal{H}$ be the statement “Every nice $\Sigma_2$–statement is true.” Is $\mathcal{H}$ consistent with ZFC?

**Note:** $\mathcal{H}$ implies there is some $\kappa$ such that $2^\kappa = \kappa^+$, and some $\kappa$ such that $2^\kappa = \kappa_+^{+7567}$, and some $\kappa$ such that $\Diamond_\kappa$ holds, and some $\kappa$ such that $\Diamond_\kappa$ fails, and so on....

“Everything possible is true at some level of the universe.”
A rather weak partial answer:

**Observation**

(A.–Venturi) ✑ is consistent with ZFC – Replacement + $\Sigma_2$–Replacement.
Thank you for your attention!