Abstract. We introduce a new method for building models of CH, together with $\Pi_2$ statements over $H(\omega_2)$, by forcing. Unlike similar constructions in the literature, our construction adds new reals, but only $\aleph_1$-many of them. Using this approach, we prove that a very strong form of the negation of Club Guessing at $\omega_1$ known as Measuring is consistent together with CH, thereby answering a well-known question of Moore. The construction works over any model of ZFC and can be described as a finite support forcing iteration with side conditions. Given a condition in our construction, the side condition consists of a countable symmetric system of models with markers and a finite undirected graph on the symmetric system. The CH-preservation is accomplished through the imposition of symmetry constraints on both the side condition and the working part as dictated by the edges in the graph.

1. Introduction

The problem of building models of consequences, at the level of $H(\omega_2)$, of classical forcing axioms in the presence of the Continuum Hypothesis (CH) has a long history, starting with Jensen’s landmark result that Suslin’s Hypothesis is compatible with CH ([9]). Much of the work in this area is due to Shelah (see [19]), with contributions also by other people (see e.g. [1], [11], [16], [10], [4] or [17]). Most of the work in the area done so far proceeds by showing that some suitable countable support iteration whose iterands are proper forcing notions not adding new reals fails to add new reals at limit stages.

There are (nontrivial) limitations to what can be achieved in this area. One conclusive example is the main result from [4], which highlights a strong ‘global’ limitation: There is no model of CH satisfying a certain mild large cardinal assumption and realizing all $\Pi_2$ statements over the structure $H(\omega_2)$ that can be forced, using proper forcing, to hold together with CH. In fact there are two $\Pi_2$ statements over $H(\omega_2)$,
each of which can be forced, using proper forcing, to hold together with CH—for one of them we need an inaccessible limit of measurable cardinals—and whose conjunction implies $2^{\aleph_0} = 2^{\aleph_1}$.

The above example is closely tied to the following well–known obstacle to not adding reals, which appears in [10] and which is more to the point in the context of this paper: Given a ladder system $\vec{C} = (C_\delta : \delta \in \text{Lim}(\omega_1))$ (i.e., each $C_\delta$ is a cofinal subset of $\delta$ of order type $\omega$), let $\text{Unif}(\vec{C})$ denote the statement that for every colouring $F : \text{Lim}(\omega_1) \rightarrow \{0, 1\}$ there is a function $G : \omega_1 \rightarrow \{0, 1\}$ with the property that for every $\delta \in \text{Lim}(\omega_1)$ there is some $\alpha < \delta$ such that $G(\xi) = F(\delta)$ for all $\xi \in C_\delta \setminus \alpha$ (where, given an ordinal $\alpha$, $\text{Lim}(\alpha)$ is the set of limit ordinals below $\alpha$). We say that $G$ uniformizes $F$ on $\vec{C}$. Given $\vec{C}$ and $F$ as above there is a natural forcing notion, let us call it $Q_{\vec{C}, F}$, for adding a uniformizing function for $F$ on $\vec{C}$ by initial segments. It takes a standard exercise to show that $Q_{\vec{C}, F}$ is proper, adds the intended uniformizing function, and does not add reals. However, any long enough iteration of forcings of the form $Q_{\vec{C}, F}$, even with a fixed $\vec{C}$, will necessarily add new reals. As a matter of fact, the existence of a ladder system $\vec{C}$ for which $\text{Unif}(\vec{C})$ holds cannot be forced together with CH in any way whatsoever, as this statement actually implies $2^{\aleph_0} = 2^{\aleph_1}$. The argument is well–known and may be found for example in [10].

In the present paper we distance ourselves from the tradition of iterating forcing without adding reals and tackle the problem of building interesting models of CH through an entirely different approach: starting with a model of CH, we build a forcing which adds new reals, although only $\aleph_1$–many of them.

In [5], a framework for building finite support forcing iterations incorporating systems of countable models as side conditions was developed (see also [2], [6], [7], and [8] for further elaborations). These iterations arise naturally in, for example, situations in which one is interested in building a forcing iteration of length $\kappa$ (where $\kappa$ is relatively long) which is proper and which, in addition, does not collapse cardinals. Much of what we will say in the next few paragraphs will probably make sense

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1We will revisit this obstacle at the end of the paper with the purpose of addressing the following question: Why do our methods work with the present application and not with the problem of forcing $\text{Unif}(\vec{C})$ (for any given $\vec{C}$)?

2As it turns out, the construction resembles a classical finite support iteration, and in fact it adds Cohen reals.

3For example if, as in [5], we want to force certain instances of the Proper Forcing Axiom (PFA) together with $2^{\aleph_0} = \kappa > \aleph_2$. 
only to readers with at least some familiarity with the framework as presented, for example, in [5].

In the situations we are referring to here, one typically aims at a construction which in fact has the $\aleph_2$-chain condition, and in order to achieve this it is natural to build the iteration in such a way that conditions be of the form $(F, \Delta)$, for $F$ a (finitely supported) $\kappa$-sequence of working parts, and with $\Delta$ being a set of models with markers, i.e., of ordered pairs $(N, \rho)$, where $N$ is a countable elementary submodel of $H(\kappa)$, possibly enhanced with some predicate $T \subseteq H(\kappa)$, and where $\rho < \kappa$. $N$ is one of the models for which we will try to ‘force’ each working part $F(\alpha)$, for every stage $\alpha \in N \cap \rho$, to be generic for the generic extension of $N$ up to that stage; thus, $\rho$ is to be seen as a ‘marker’ that tells us up to which point is $N$ ‘active’ as a side condition.

In order for the construction to have the $\aleph_2$-chain condition and to be proper, it is often necessary to start from a model of $\text{CH}$ and require that the domain of $\Delta$ be a set of models with suitable symmetry properties. We call (finite) sets of models having these properties $T$-symmetric systems (for a fixed $T \subseteq H(\kappa)$). One of these properties, and the one on which we will focus our attention in a moment, is the following: In a $T$-symmetric system $\mathcal{N}$, if $N$ and $N'$ are both in $\mathcal{N}$ and $N \cap \omega_1 = N' \cap \omega_1$, then there is a (unique) isomorphism $\Psi_{N,N'}$ between the structures $(N; \in, T, N \cap N)$ and $(N'; \in, T, \mathcal{N} \cap N')$ which, moreover, is the identity on $N \cap N'$.

At this point one could as well take a step back and analyse the pure side condition forcing $P_0$ by itself. This forcing $P_0$, which we can naturally see as the first stage of our construction, consists of all finite $T$-symmetric systems of submodels, ordered by reverse inclusion. $P_0$ first appeared in the literature in [21]. It is a relatively well-known fact, and was noted in [7],\(^4\) that forcing with $P_0$ adds Cohen reals, although not too many; in fact it adds exactly $\aleph_1$-many of them. This may be somewhat surprising given that $P_0$ adds, by finite approximations, a new rather large object (a symmetric system covering all of $H(\kappa)^V$).\(^5\) The argument for this is contained in the proof of Lemma 3.15 from the present paper, but it will nonetheless be convenient at this point to sketch it here.

We assume, towards a contradiction, that there is a sequence $(\dot{r}_\nu)_{\nu < \omega_2}$ of $P_0$-names which some condition $\mathcal{N}$ forces to be distinct subsets of $\omega$.

\(^4\)See also [15].

\(^5\)Incidentally, $P_0$ is in fact strongly proper, and so each new real it adds is in fact contained in an extension of $V$ by some Cohen real. The preservation of $\text{CH}$ by $P_0$ was exploited in [13].
Without loss of generality we can take each \( \dot{r}_\nu \) to be a member of \( H(\kappa) \).

For each \( \nu \) we can pick \( N_\nu \) to be a sufficiently correct countable model containing all relevant objects, which in this case includes \( \mathcal{N} \) and \( \dot{r}_\nu \).

As CH holds, we may find distinct indices \( \nu \) and \( \nu' \) such that there is a unique isomorphism \( \Psi_{N_\nu, N_{\nu'}} \) between the structures \( (N_\nu; \in, T^*, \mathcal{N}, \dot{r}_\nu) \) and \( (N_{\nu'}; \in, T^*, \mathcal{N}, \dot{r}_{\nu'}) \) fixing \( N_\nu \cap N_{\nu'} \), where \( T^* \subseteq H(\kappa) \) codes the satisfaction relation for \( (H(\kappa); \in, T) \). But then \( \mathcal{N}^* = \mathcal{N} \cup \{ N_\nu, N_{\nu'} \} \) is a condition in \( \mathcal{P}_0 \) forcing that \( \dot{r}_\nu = \dot{r}_{\nu'} \). The point is that if \( n \in \omega \) and \( \mathcal{N}^* \) is any condition extending \( \mathcal{N}^* \) and forcing \( n \in \dot{r}_\nu \), then \( \mathcal{N}^* \) is in fact compatible with a condition \( \mathcal{M} \subseteq N_\nu \) forcing the same thing. This is true since \( \mathcal{N}^* \) is an \( (N_\nu, \mathcal{P}_0) \)-generic condition. But then \( \Psi_{N_\nu, N_{\nu'}}(\mathcal{M}) \) is a condition forcing \( n \in \Psi_{N_{\nu'}, N_{\nu'}}(\dot{r}_{\nu'}) \) (since \( \mathcal{P}_0 \) is definable in \( (H(\kappa); \in, T^*) \) without parameters). Finally, if \( \mathcal{N}'' \) is a common extension of \( \mathcal{N}' \) and \( \mathcal{M} \), then \( \mathcal{N}'' \) forces also that \( n \in \dot{r}_{\nu'} \), since it extends \( \Psi_{N_\nu, N_{\nu'}}(\mathcal{M}) \) as \( \Psi_{N_\nu, N_{\nu'}}(\mathcal{M}) \subseteq \mathcal{N}'' \) by the symmetry requirement.

\( \mathcal{P}_0 \) has received some attention in the literature. For example, Todor-čević proved that \( \mathcal{P}_0 \) adds a Kurepa tree (s. [15]). Also, [15] presents a mild variant of \( \mathcal{P}_0 \) which not only preserves CH but actually forces \( \diamond \).

The iterations with symmetric systems of models as side conditions that we were referring to before do not preserve \( CH \), and in fact they force \( 2^{\aleph_0} = \kappa > \aleph_1 \). The reason is of course that there are no symmetry requirements on the working parts. Hence, even if the first stage of the iterations—which is, essentially, \( \mathcal{P}_0 \)—preserves \( CH \), the iterations are in fact designed to add new reals at all later (successor) stages.

Something one may naturally envision at this point is the possibility to build a suitable forcing with systems of models (with markers) as side conditions while strengthening the symmetry constraints, so as to make them apply not only to the side condition part of the forcing but also to the working parts; one would hope to exploit the above idea in order to show that the forcing thus constructed preserves \( CH \), and would of course like to be able to do that while at the same time forcing some interesting statement. In the present paper we implement this idea by proving the relative consistency with \( CH \) of a very strong form of the failure of Club Guessing at \( \omega_1 \) known as \textbf{Measuring} (see [10]) that follows from PFA.

\textbf{Definition 1.1.} \textbf{Measuring} holds if and only if for every sequence \( \bar{C} = (C_\delta : \delta \in \omega_1) \), if each \( C_\delta \) is a closed subset of \( \delta \) in the order topology, then there is a club \( C \subseteq \omega_1 \) such that for every \( \delta \in C \) there is some \( \alpha < \delta \) such that either

\begin{itemize}
  \item \( (C \cap \delta) \setminus \alpha \subseteq C_\delta \), or
\end{itemize}
• \((C \setminus \alpha) \cap C_\delta = \emptyset\).

In the above definition, we say that \(C\) measures \(\bar{C}\). Measuring is of course equivalent to its restriction to club-sequences \(\bar{C}\) on \(\omega_1\), i.e., to sequences of the form \(\bar{C} = (C_\delta : \delta \in \text{Lim}(\omega_1))\), where each \(C_\delta\) is a club of \(\delta\). It is also not difficult to see that Measuring can be rephrased as the assertion that the algebra \(\mathcal{P}(\omega_1)/\text{NS}_{\omega_1}\)—where \(\text{NS}_{\omega_1}\) denotes the nonstationary ideal on \(\omega_1\)—forces that \(C_{\omega_1}^V\) is a base for an ultrafilter on the Boolean subalgebra of \(\mathcal{P}(\omega_1^V)\) generated by the closed sets as computed in the generic ultrapower \(M = V/\dot{G}\), where \(C_{\omega_1}^V\) denotes the club filter on \(\omega_1\) in \(V\).

In the present paper we implement the aforementioned approach by showing that Measuring is indeed a statement which can be forced adding new reals while, nevertheless, preserving CH. Our main theorem is the following.

**Theorem 1.2.** \((\text{CH} + 2^{\omega_1} = \aleph_2)\) There is a partial order \(\mathcal{P}\) with the following properties.

1. \(|\mathcal{P}| = \aleph_2\)
2. \(\mathcal{P}\) is proper and \(\aleph_2\)-Knaster.
3. \(\mathcal{P}\) forces the following statements.
   a. Measuring
   b. CH

Theorem 1.2 answers a question of Moore, who asked if Measuring is compatible with CH (see [10] or [18]). There are natural proper forcing notions for adding a club of \(\omega_1\) measuring a given club-sequence by countable approximations. These forcings do not add new reals, but it is not known whether their countable support iterations also (consistently) have this property; indeed, for all is known, these measuring forcings may always fall outside all of the currently available iteration schemes for iterating proper forcing without adding reals (s. [10]). We should point out that the strongest failures of Club Guessing at \(\omega_1\) known to be within reach of the current forcing iteration methods for producing models of CH without adding new reals (s. [20]) seem to be only in the region of the negation of weak Club Guessing at \(\omega_1\), \(\neg\text{WCG}\), which is the statement that for every ladder system \((C_\delta : \delta \in \text{Lim}(\omega_1))\)

\(^6\)On the other hand, the limit of such a countable support iteration will be \(\omega\)-bounding (i.e., every function \(f \in {}^\omega \omega\) in the extension is bounded by some such function from the ground model). In particular, if CH—or even \(\delta = \omega_1\)—holds in the ground model, then \(\delta = \omega_1\) will of course hold in the extension (where \(\delta\) is the minimal cardinality of a family \(\mathcal{F} \subseteq {}^\omega \omega\) with the property that for every \(g \in {}^\omega \omega\) there is some \(f \in \mathcal{F}\) such that \(g(n) < f(n)\) for co-boundedly many \(n < \omega\).
there is a club $C \subseteq \omega_1$ which has finite intersection with each $C_\delta$. On the other hand, Measuring certainly implies $\neg$WCG.\textsuperscript{7}

In [8]) we consider a cardinal–preserving iteration with finite supports and with symmetric systems of models as side conditions for forcing Measuring together with $2^{\aleph_0} = \kappa$, with $\kappa$ being arbitrarily large. It turns out that such a construction can be modified in such a way as to yield a model of Measuring together with CH. This modification is the content of the present paper. As we will see, it involves strong symmetry constraints on both working parts and side condition as dictated by suitable ‘edges’ $\langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle$ of models with markers coming from the side condition. Also, for technical reasons our present construction involves side conditions consisting of infinite symmetric systems of models, only finitely many of which have nonzero marker.

Rather than delving into more details here, we refer the reader to the actual construction in Section 2.\textsuperscript{8} We conclude this introduction with some observations on ways to extend Measuring and an open problem.

1.1. Some observations and an open problem. It is not hard to see that Measuring is equivalent to the statement that if $(C_\delta : \delta \in \text{Lim}(\omega_1))$ is such that each $C_\delta$ is a countable collection of closed subsets of $\omega_1$, then there is a club of $\omega_1$ measuring all members of $C_\delta$ for each $\delta$. We may thus consider the following family of strengthenings of Measuring.

**Definition 1.3.** Given a cardinal $\kappa$, Meas$_\kappa$ holds if and only if for every family $\mathcal{C}$ consisting of closed subsets of $\omega_1$ and such that $|\mathcal{C}| \leq \kappa$ there is a club $C \subseteq \omega_1$ with the property that for every $D \in \mathcal{C}$ and every $\delta \in C$ there is some $\alpha < \delta$ such that either

- $(C \cap \delta) \setminus \alpha \subseteq D$, or
- $(C \cap \delta) \setminus \alpha \cap D = \emptyset$.

Meas$_{\aleph_0}$ is trivially true in ZFC. Also, it is clear that Meas$_\kappa$ implies Meas$_\lambda$ whenever $\lambda < \kappa$, and that Meas$_{\aleph_1}$ implies Measuring.

Recall that the splitting number, $s$, is the minimal cardinality of a splitting family, i.e., of a collection $\mathcal{X} \subseteq [\omega]^{\aleph_0}$ such that for every $Y \in [\omega]^{\aleph_0}$ there is some $X \in \mathcal{X}$ such that $X \cap Y$ and $Y \setminus X$ are both infinite.

\textsuperscript{7} Indeed, suppose $(C_\delta : \delta \in \text{Lim}(\omega_1))$ is a ladder system and $D \subseteq \omega_1$ is a club measuring it. Then every limit point $\delta \in D$ of limit points of $D$ is such that $D \cap C_\delta$ is bounded in $\delta$ since no tail of $D \cap \delta$ can possibly be contained in $C_\delta$ as $C_\delta$ has order type only $\omega$.

\textsuperscript{8} We write some more specific motivation right after the construction.
In the proof of Fact 1.4, if \((C_\delta : \delta \in \text{Lim}(\omega_1))\) is a ladder system on \(\omega_1\), we write \((C_\delta(n))_{n < \omega}\) to denote the strictly increasing enumeration of \(C_\delta\). Also, here and throughout the paper, \([\alpha, \beta] = \{\xi \in \text{Ord} : \alpha \leq \xi \leq \beta\}\) for all ordinals \(\alpha \leq \beta\).

**Fact 1.4.** \(\text{Meas} \) is false.

**Proof.** Let \(\mathcal{X} \subseteq [\omega]^{\aleph_0}\) be a splitting family. Let \((C_\delta)_{\delta \in \text{Lim}(\omega)}\) be a ladder system on \(\omega_1\) such that \(C_\delta(n)\) is a successor ordinal for each \(\delta \in \text{Lim}(\omega_1)\) and \(n < \omega\), and let \(\mathcal{C}\) be the collection of all sets of the form

\[
Z_\delta^X = \bigcup \{[C_\delta(n), C_\delta(n + 1)) : n \in X\} \cup \{\delta\}
\]

for some \(\delta \in \text{Lim}(\omega_1)\) and \(X \in \mathcal{X}\). Let \(D\) be a club of \(\omega_1\), let \(\delta < \omega_1\) be a limit point of \(D\), and let

\[
Y = \{n < \omega : [C_\delta(n), C_\delta(n + 1)) \cap D \neq \emptyset\}
\]

Let \(X \in \mathcal{X}\) be such that \(X \cap Y\) and \(Y \setminus X\) are infinite. Then \(Z_\delta^X \cap D\) and \(D \setminus Z_\delta^X\) are both cofinal in \(\delta\). Hence, \(D\) does not measure \(\mathcal{C}\). \(\square\)

The following is proved in recent joint work of the first author with John Krueger.

**Theorem 1.5.** ([3]) \(\text{Meas}^*_\omega\) can be forced over any model of ZFC and follows from \(\text{BPFA}\).

Another natural way to strengthen \(\text{Measuring}\) is to allow, in the sequence to be measured, not just closed sets, but also sets of higher complexity. The version of \(\text{Measuring}\) where one considers sequences \(\mathcal{X} = (X_\delta : \delta \in \text{Lim}(\omega_1))\), with each \(X_\delta\) an open subset of \(\delta\) in the order topology, is of course equivalent to \(\text{Measuring}\). A natural next step would therefore be to consider sequences in which each \(X_\delta\) is some countable union of closed sets. This is of course the same as allowing each \(X_\delta\) to be an arbitrary subset of \(\delta\). Let us call the corresponding statement \(\text{Measuring}^*\):

**Definition 1.6.** \(\text{Measuring}^*\) holds if and only if for every sequence \(\mathcal{X} = (X_\delta : \delta \in \text{Lim}(\omega_1))\), if \(X_\delta \subseteq \delta\) for all \(\delta\), then there is some club \(C \subseteq \omega_1\) such that for every \(\delta \in C\), a tail of \(C \cap \delta\) is either contained in or disjoint from \(X_\delta\).

It is easy to see that \(\text{Measuring}^*\) is false in ZFC. In fact, given a stationary and co-stationary \(S \subseteq \omega_1\), there is no club of \(\omega_1\) measuring \(\mathcal{X} = (S \cap \delta : \delta \in \text{Lim}(\omega_1))\). In fact, if \(C\) is any club of \(\omega_1\), then both \(C \cap S \cap \delta\) and \((C \cap \delta) \setminus S\) are cofinal subsets of \(\delta\) for each \(\delta\) in the club of limit points in \(\omega_1\) of both \(C \cap S\) and \(C \setminus S\).
The status of Measuring\(^*\) is more interesting in the absence of the Axiom of Choice. Let \(\mathcal{C}_{\omega_1} = \{ X \subseteq \omega_1 : C \subseteq X \text{ for some club } C \text{ of } \omega_1 \}\).

**Observation 1.1.** \((ZF + \mathcal{C}_{\omega_1} \text{ is a normal filter on } \omega_1)\) Suppose \(\vec{X} = (X_\delta : \delta \in \text{Lim}(\omega_1))\) is such that

1. \(X_\delta \subseteq \delta \text{ for each } \delta\).
2. For each club \(C \subseteq \omega_1\),
   - (a) there is some \(\delta \in C\) such that \(C \cap X_\delta \neq \emptyset\), and
   - (b) there is some \(\delta \in C\) such that \((C \cap \delta) \setminus X_\delta \neq \emptyset\).

Then there is a stationary and co-stationary subset of \(\omega_1\) definable from \(\vec{X}\).

**Proof.** We have two possible cases. The first case is when for all \(\alpha < \omega_1\), either

- \(W_0^\alpha = \{ \delta < \omega_1 : \alpha \notin X_\delta \}\) is in \(\mathcal{C}_{\omega_1}\), or
- \(W_1^\alpha = \{ \delta < \omega_1 : \alpha \in X_\delta \}\) is in \(\mathcal{C}_{\omega_1}\).

For each \(\alpha < \omega_1\) let \(W_\alpha\) be \(W_\epsilon^\alpha\) for the unique \(\epsilon \in \{0, 1\}\) such that \(W_\epsilon^\alpha \in \mathcal{C}_{\omega_1}\), and let \(W^* = \Delta_{\delta < \omega_1} W_\alpha \in \mathcal{C}_{\omega_1}\). Then \(X_{\delta_0} = X_{\delta_1} \cap \delta_0\) for all \(\delta_0 < \delta_1\) in \(W^*\). It then follows, by (2), that \(S = \bigcup_{\delta \in W^*} X_\delta\), which of course is definable from \(\vec{C}\), is a stationary and co-stationary subset of \(\omega_1\). Indeed, suppose \(C \subseteq \omega_1\) is a club, and let us fix a club \(D \subseteq W^*\). There is then some \(\delta \in C \cap D\) and some \(\alpha \in C \cap D \cap X_\delta\). But then \(\alpha \in S\) since \(\delta \in W^*\) and \(\alpha \in W^* \cap X_\delta\). There is also some \(\delta \in C \cap D\) and some \(\alpha \in C \cap D\) such that \(\alpha \notin X_\delta\), which implies that \(\alpha \notin S\) by a symmetrical argument, using the fact that \(X_{\delta_0} = X_{\delta_1} \cap \delta_0\) for all \(\delta_0 < \delta_1\) in \(W^*\).

The second possible case is that there is some \(\alpha < \omega_1\) with the property that both \(W_0^\alpha\) and \(W_1^\alpha\) are stationary subsets of \(\omega_1\). But now we can let \(S\) be \(W_0^\alpha\), where \(\alpha\) is first such that \(W_0^\alpha\) is stationary and co-stationary.

It is worth comparing the above observation with Solovay’s classic result that an \(\omega_1\)-sequence of pairwise disjoint stationary subsets of \(\omega_1\) is definable from any given ladder system on \(\omega_1\) (working in the same theory).

**Corollary 1.7.** \((ZF + \mathcal{C}_{\omega_1} \text{ is a normal filter on } \omega_1)\) The following are equivalent.

1. \(\mathcal{C}_{\omega_1}\) is an ultrafilter on \(\omega_1\).
2. Measuring\(^*\)
3. For every sequence \((X_\delta : \delta \in \text{Lim}(\omega_1))\), if \(X_\delta \subseteq \delta\) for each \(\delta\), then there is a club \(C \subseteq \omega_1\) such that either
   - \(C \cap \delta \subseteq X_\delta\) for every \(\delta \in C\), or
Few new reals

\[ C \cap X_\delta = \emptyset \text{ for every } \delta \in C. \]

Proof. (3) trivially implies (2), and by the observation (1) implies (3). Finally, to see that (2) implies (1), note that the argument right after the definition of Measuring* uses only ZF together with the regularity of \( \omega_1 \) and the negation of (1). \( \square \)

In particular, the strong form of Measuring* given by (3) in the above observation follows from ZF together with the Axiom of Determinacy.

Finally, and back in ZFC, the following question, suggested by Moore, aims at addressing the issue whether or not adding new reals is a necessary feature of any successful approach to forcing Measuring + CH.

**Question 1.8.** Does Measuring imply that there are non-constructible reals?

Much of the notation used in this paper follows the standards set forth in [12] and [14]. Other, less standard, pieces of notation will be introduced as needed. The rest of the paper is structured as follows. In Section 2 we present our forcing construction, and in Section 3 we prove the relevant facts about this construction which together yield the proof of Theorem 1.2. We conclude the paper with Section 4, which contains some remarks on why our construction cannot possibly be adapted to force Unif(\( \tilde{C} \)) for any ladder system \( \tilde{C} \) (which, as we already mentioned, is well-known to be incompatible with CH), and on the (closely related) obstacles towards building models of reasonable forcing axioms together with CH using the present approach.

2. THE CONSTRUCTION

In this section we define the forcing witnessing Theorem 1.2. Let us assume CH and \( 2^{\aleph_1} = \aleph_2 \).

We start out by fixing some pieces of notation that will be used throughout the paper: If \( N \) is a set such that \( N \cap \omega_1 \in \omega_1 \), \( \delta_N \) denotes \( N \cap \omega_1 \). \( \delta_N \) is also called the height of \( N \). If \( T \subseteq H(\omega_2) \) and \( N \subseteq H(\omega_2) \), we will write \( (N,T) \) as short-hand for \( (N,T \cap N) \). Also, if \( N_0 \) and \( N_1 \) are \( \in \)-isomorphic models, we refer to the unique \( \in \)-isomorphism \( \Psi : (N_0; \in) \rightarrow (N_1; \in) \) as \( \Psi_{N_0,N_1} \). This isomorphism \( \Psi_{N_0,N_1} \) extends naturally to a unique isomorphism

\[ \overline{\Psi} : (\text{cl}(N_0); \in) \rightarrow (\text{cl}(N_1); \in), \]

where, given a set \( X \), cl(\( X \)) denotes \( X \cup X \cap \text{Ord} \), and where \( \overline{X \cap \text{Ord}} \) stands for the closure of \( X \cap \text{Ord} \) in the order topology. We will denote this unique extension \( \overline{\Psi} \) by \( \overline{\Psi}_{N_0,N_1} \).
Let us fix a function $\Phi : \omega_2 \to H(\omega_2)$ such that $\Phi^{-1}(x)$ is unbounded in $\omega_2$ for all $x \in H(\omega_2)$. Note that $\Phi$ exists by $2^{\aleph_1} = \aleph_2$. We let $(T_\beta)_{\beta < \omega_2}$ be a sequence of predicate of $H(\omega_2)$ defined recursively by letting $T_\beta$ be as follows.

- If $\beta$ is a limit ordinal, then $T_\beta = \bigcup_{\alpha < \beta} T_\alpha$.
- If $\beta = \alpha + 1$, then $T_\beta = T_\alpha \cup \{(\alpha, x) : x \in S_\alpha\}$, where $S_\alpha$ is the satisfaction predicate for the structure $(H(\omega_2); \in, \Phi, T_\alpha)$. 

Also, given $\beta < \omega_2$, we let $T_\beta$ be the collection of countable elementary submodels of $(H(\omega_2); \in, T_\beta)$.

The following fact is immediate.

**Fact 2.1.** Let $\alpha < \beta \leq \omega_2$.

1. If $N \in T_\beta$ and $\alpha \in N$, then $N \in T_\alpha$.
2. If $N, N' \in T_\beta$, $\Psi : (N; \in, T_\beta) \to (N'; \in, T_\beta)$ is an isomorphism, and $M \in N \cap T_\beta$, then $\Psi(M) \in T_\beta$.

We will call ordered pairs $(N, \rho)$, where

- $N$ is a countable elementary submodel of $H(\omega_2)$,
- $\rho$ is in the closure of $N \cap \omega_2$ in the order topology, and
- for every $\alpha \in N \cap \rho$, $N \in T_\alpha$,

*models with marker.*

In our forcing construction, we will use models with markers $(N, \rho)$ in a crucial way. The marker $\rho$ will tell us for which stages is $N$ to be seen as ‘active’ within a given condition in the forcing (namely all stages in $N \cap \rho$).

Obviously, if $(M, \rho)$ and $(N, \rho')$ are models with markers, then $M \in N$ if and only if $(M, \rho) \in N$.

We will need the following slight variation of the notion of symmetric system from [5].

**Definition 2.2.** Let $P \subseteq H(\omega_2)$, and let $\mathcal{N}$ be a finite or countable collection of countable subsets of $H(\omega_2)$. We say that $\mathcal{N}$ is a $P$-symmetric system if and only if the following holds.

1. For every $N \in \mathcal{N}$, $(N; \in, P)$ is an elementary substructure of $(H(\omega_2); \in, T)$. 

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9The only difference between our present notion of symmetric system and the notion defined in [5] is that the symmetric systems from [5] are finite, whereas in the present situation they may also be infinite.
(2) Given $N_0$ and $N_1$ in $\mathcal{N}$, if $\delta_{N_0} = \delta_{N_1}$, then there is a unique isomorphism

$$\Psi_{N_0,N_1} : (N_0; \in, P) \to (N_1; \in, P)$$

Furthermore, $\Psi_{N_0,N_1}$ is the identity on $N_0 \cap N_1$.

(3) For all $N$ and $M$ in $\mathcal{N}$, if $\delta_M < \delta_N$, then there is some $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

(4) For all $N_0$, $N_1$, $M \in \mathcal{N}$, if $M \in N_0$ and $\delta_{N_0} = \delta_{N_1}$, then $\Psi_{N_0,N_1}(M) \in \mathcal{N}$.

If $P = \emptyset$, we refer to $P$-symmetric systems simply as symmetric systems.

The above definitions obviously make sense also in contexts where we consider collections of countable elementary submodels of $H(\kappa)$ for $\kappa > \omega_2$. In any case, it is convenient to point out that when $\kappa = \omega_2$, $2^{\omega_1} = \aleph_3$, and the relevant predicates code enough information (as is the case in the present situation with the predicate $\Phi$), the requirement that the natural extension of the unique isomorphism between isomorphic models $N_0$ and $N_1$ be the identity on $\text{cl}(N_0) \cap \text{cl}(N_1)$ is superfluous, as this is an immediate consequence of the following fact.

**Fact 2.3.** Let $\bar{g} = (g_\beta : \beta < \omega_2)$, where $g_\beta : |\beta| \to \beta$ is a surjection for each $\beta < \omega_2$, and suppose $N_0$ and $N_1$ are countable subsets of $H(\omega_2)$ such that $(N_0; \in, \bar{g})$, $(N_1; \in, \bar{g}) \preccurlyeq (H(\omega_2); \in, \bar{g})$ and $\delta_{N_0} = \delta_{N_1}$. Then $N_0 \cap N_1 \cap \omega_2$ is an initial segment of both $N_0 \cap \omega_2$ and $N_1 \cap \omega_2$.

**Proof.** Let $\beta \in N_0 \cap N_1 \cap \omega_2$ and let $\alpha \in \beta \cap N_0$. We want to see that $\alpha \in N_1$. For this, let $\xi \in |\beta| \cap N_0$ be such that $g_\beta(\xi) = \alpha$. Since $\delta_{N_0} = \delta_{N_1}$, it follows that $\xi \in N_1$. Since of course $|\beta| \in N_1$, we conclude that $\alpha = g_\beta(\xi) \in N_1$. $\square$

The following amalgamation lemmas are proved in [5] for finite symmetric systems, and exactly the same proofs work for the present notion.

**Lemma 2.4.** Let $P \subseteq H(\omega_2)$, let $\mathcal{N}$ be a $P$-symmetric system, and let $N \in \mathcal{N}$. If $\mathcal{M} \in N$ is a $P$-symmetric system such that $\mathcal{M} \supseteq \mathcal{N} \cap N$, then

$$\mathcal{N} \cup \bigcup \{\Psi_{N,N'} : \mathcal{M} : N' \in \mathcal{N}, \delta_{N'} = \delta_N\}$$

is the $\subseteq$-minimal $P$-symmetric system $\mathcal{W}$ such that $\mathcal{N} \cup \mathcal{M} \subseteq \mathcal{W}$.

We will denote the symmetric system

$$\mathcal{N} \cup \bigcup \{\Psi_{N,N'} : \mathcal{M} : N' \in \mathcal{N}, \delta_{N'} = \delta_N\}$$

in the statement of Lemma 2.4 by $\mathcal{A}(\mathcal{N}, \mathcal{M}, N)$. 

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Given $P \subseteq H(\omega_2)$ and $P$-symmetric systems $\mathcal{N}_0$ and $\mathcal{N}_1$, we write $\mathcal{N}_0 \cong \mathcal{N}_1$ iff $|\mathcal{N}_0| = |\mathcal{N}_1| = \mu$, for some $\mu \leq \aleph_0$, and there are enumerations $(N_0^i)_{i<\mu}$ and $(N_1^i)_{i<\mu}$ of $\mathcal{N}_0$ and $\mathcal{N}_1$, respectively, for which there is an isomorphism

$$\Psi : (\bigcup N_0; \varepsilon, P, N_0^i)_{i<\mu} \rightarrow (\bigcup N_1; \varepsilon, P, N_1^i)_{i<\mu}$$

which is the identity on $(\bigcup \mathcal{N}_0) \cap (\bigcup \mathcal{N}_1)$.

**Lemma 2.5.** Let $P \subseteq H(\omega_2)$, let $\mathcal{N}_0$ and $\mathcal{N}_1$ be $P$-symmetric systems, and suppose $\mathcal{N}_0 \cong \mathcal{N}_1$. Then $\mathcal{N}_0 \cup \mathcal{N}_1$ is a $P$-symmetric system.

Given a predicate $P \subseteq H(\omega_2)$ and a symmetric system $\mathcal{N}$ and given $M, N \in \mathcal{N}$ such that $\delta_M \leq \delta_N$, we can canonically associate to $M$ and $N$ a function $\Psi^N_{M,N} : M \rightarrow N$. This is the unique function $\Psi : M \rightarrow N$ such that

- $\Psi = \Psi^N_{M,N}$ if $\delta_M = \delta_N$, and
- if $\delta_M < \delta_N$, $\Psi = \Psi^N_{N',N} \upharpoonright M$ for some (equivalently, any) $N' \in \mathcal{N}$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

Notice that $\Psi^N_{M,N}$ is an elementary embedding from $(M; \varepsilon, P)$ into $(N; \varepsilon, P)$ and that if $M_0, M_1, N \in \mathcal{N}$ are such that $\delta_{M_1} \leq \delta_{M_0} \leq \delta_N$, then $\Psi^N_{M_1, N} = \Psi^N_{M_0, M_1} \circ \Psi^N_{M_0, N}$.

We will be making extensive use of the following construct: Given models with markers $(N_0, \rho_0)$ and $(N_1, \rho_1)$, we will say that the ordered pair $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ is an edge in case $(N_0; \varepsilon) \cong (N_1; \varepsilon)$.

We will say that a set $\tau$ of edges is from a set $\Delta$ of models with markers in case for every edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau$ both $(N_0, \rho_0)$ and $(N_1, \rho_1)$ are in $\Delta$. Also, we will call a set of edges symmetric if for every edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ it holds that $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau$ if and only if $\langle (N_1, \rho_1), (N_0, \rho_0) \rangle \in \tau$.

We will call a function $F$ relevant if $\text{dom}(F) \subseteq [\omega_2]^{<\omega}$ and for every $\alpha \in \text{dom}(F)$, $F(\alpha) = (I_\alpha, b_\alpha)$, where

- $I_\alpha$ is a finite collection of pairwise disjoint intervals $[\delta_0, \delta_1]$, where $\delta_0 \leq \delta_1 < \omega_1$,
- $b_\alpha$ is a function such that $\text{dom}(b_\alpha) \subseteq \{ \text{min}(I) : I \in I_\alpha \}$ and such that $b_\alpha(\delta) < \delta$ for every $\delta \in \text{dom}(b_\alpha)$.

In the above situation, we will often refer to $I_\alpha$ and $b_\alpha$ as, respectively, $I^\alpha$ and $b^\alpha$. If $\alpha \notin \text{dom}(F)$, then we let $I^\alpha$ and $b^\alpha$ denote the empty set.

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10Due to the fact that $\Psi^N_{N',N''}(M) = M$ for all $N'$, $N'' \in \mathcal{N}$ of the same height as $N$ such that $M \in N' \cap N''$, the definition of $\Psi^N_{M,N}$ in this case does not depend on the $N'$ chosen.
Throughout the paper, given a triple \( q = (F, \Delta, \tau) \), where \( F \) is a relevant function, \( \Delta \) is a set of models with markers, and \( \tau \) is a set of edges from \( \Delta \), we will often denote \( F, \Delta \) and \( \tau \) by, respectively, \( F_q, \Delta_q \) and \( \tau_q \). Also, given \( \alpha \in \text{dom}(F_q) \), we will denote \( \mathcal{I}_\alpha^{F_q} \) and \( b_\alpha^{F_q} \) by, respectively, \( \mathcal{I}_\alpha^q \) and \( b_\alpha^q \).

If \( \Delta \) is a collection of models with markers and \( \beta < \omega_2 \) is an ordinal, \( \mathcal{N}_\beta^\Delta \) denotes the set
\[
\{ N : (N, \rho) \in \Delta, \ \beta \in N \cap (\rho + 1) \}
\]
Note that \( \mathcal{N}_0^\Delta \) is nothing but \( \text{dom}(\Delta) \). If \( q = (F_q, \Delta_q, \tau_q) \), where \( F_q, \Delta_q \) and \( \tau_q \) are as above, we let \( \mathcal{N}_\beta^q \) stand for \( \mathcal{N}_\beta^\Delta \). Also, if \( G \) is a set of triples \( q = (F_q, \Delta_q, \tau_q) \) as above, then
\[
\mathcal{N}_\beta^G = \bigcup \{ \mathcal{N}_\beta^q : q \in G \}
\]

If \( N \) is a set and \( \gamma \) is an ordinal, we let \( \gamma_N \) be the highest ordinal \( \tau \in N \) such that \( \tau \leq \gamma \). Given an ordinal \( \alpha \) and a set \( \Delta \) of models with markers, we let \( \Delta|_{\alpha} \) be the set of models with markers of the form
\[
(N, \min\{\alpha, \rho\}_N),
\]
where \( (N, \rho) \in \Delta \). Also, if \( \tau \) is a set of edges, we let \( \tau|_{\alpha} \) be the set of edges of the form
\[
((\langle N_0, \min\{\alpha, \rho_0\}\rangle_{N_0}, (N_1, \min\{\alpha, \rho_1\})_{N_1}),
\]
where \( \langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau \).

Whenever \( f_0, \ldots, f_n \) is a finite set of functions, we will write \( f_n \circ \ldots \circ f_0 \) to denote the (partially defined) function on \( \text{dom}(f_0) \) given by the expression \( (f_n \circ \ldots \circ f_0)(x) \) whenever this expression is defined; in other words, \( f_n \circ \ldots \circ f_0 \) is the function \( f \) with domain the set of \( x \in \text{dom}(f_0) \) such that for each \( i < n \), \( (f_i \circ \ldots \circ f_0)(x) \in \text{dom}(f_{i+1}) \), and such that for every \( x \in \text{dom}(f) \), \( f(x) = (f_n \circ \ldots \circ f_0)(x) \).

Given a sequence \( \vec{E} = \langle (\langle N_0^i, \rho_0^i \rangle, (N_1^i, \rho_1^i) \rangle \ | \ i < n \rangle \) of edges, we will tend to denote the function
\[
\Psi_{N_0^{n-1}, N_1^{n-1}} \circ \ldots \circ \Psi_{N_0^0, N_1^0}
\]
by \( \Psi_{\vec{E}} \). If \( \vec{E} \) is the empty sequence, we let \( \Psi_{\vec{E}} \) be the identity function. Given a set \( \tau \) of edges, a sequence \( \vec{E} = \langle (\langle N_0^i, \rho_0^i \rangle, (N_1^i, \rho_1^i) \rangle \ | \ i < n \rangle \) of edges from \( \tau \), and \( x \in N_0^0 \), we will call \( \langle \vec{E}, x \rangle \) a \( \tau \)-thread in case \( x \in \text{dom}(\Psi_{\vec{E}}) \). We will occasionally just say thread when \( \tau \) is not relevant. If \( y \in H(\omega_2) \) and \( \alpha < \omega_2 \), we call a \( \tau \)-thread \( \langle \vec{E}, (y, \alpha) \rangle \) correct in case
\[
1. \ \alpha < \rho_0^0,
\]
(2) \( \Psi_\mathcal{E}(\alpha) < \rho_1^{n-1} \), and
(3) for every formula \( \varphi(x) \) in the language for \( (H(\omega_2); \in, T_\alpha) \) and every \( a \in \text{dom}(\Psi_\mathcal{E}) \),
\[
(N_0^0; \in, T_\alpha) \models \varphi(a)
\]
if and only if
\[
(N_{n-1}^1; \in, T_{\Psi_\mathcal{E}(a)}) \models \varphi(\Psi_\mathcal{E}(a))
\]

Given a sequence \( \mathcal{E} = ((N_i^0, \rho_0), (N_i^1, \rho_1)) \mid i < n \) of edges, we will denote by \( \mathcal{C}_\mathcal{E} \) the set of ordinals \( \alpha \) such that \( \langle \mathcal{E}, (\emptyset, \alpha) \rangle \) is a correct thread.

We will say that a set \( \tau \) of edges is \textit{closed under copying} in case for every sequence \( \mathcal{E} \) of edges coming from \( \tau \) and every edge \( \epsilon = ((M_0, \rho_0), (M_1, \rho_1)) \in \tau \), if \( \epsilon \in \text{dom}(\Psi_\mathcal{E}), \xi_0 \in \mathcal{C}_\mathcal{E} \cap \text{cl}(M_0) \cap (\rho_0 + 1) \) and \( \xi_1 \in \mathcal{C}_\mathcal{E} \cap \text{cl}(M_1) \cap (\rho_1 + 1) \), then there are \( \rho_0 \geq \Psi_\mathcal{E}(\xi_0) \) and \( \rho_1 \geq \Psi_\mathcal{E}(\xi_1) \) such that
\[
(\Psi_\mathcal{E}(M_0), \rho_0), (\Psi_\mathcal{E}(M_1), \rho_1)) \in \tau
\]

If \( \tau \) is a set of edges and \( F \) is a relevant function \( F \), we will say that \( F \) is \textit{closed under copying via} \( \tau \) in case for every sequence \( \mathcal{E} = ((N_i^0, \rho_0), (N_i^1, \rho_1)) \mid i < n \) of edges coming from \( \tau \) and every \( \alpha \in \text{dom}(F) \cap \mathcal{C}_\mathcal{E}, \bar{\alpha} := \Psi_\mathcal{E}(\alpha) \) is in \( \text{dom}(F) \) and the following holds, where \( \delta = \min\{\delta_{N_i^0} : i < n\} \).

- \( \{I \in \mathcal{I}_\bar{\alpha}^F : \min(I) < \delta\} = \{I \in \mathcal{I}_\alpha^F : \min(I) < \delta\} \)
- \( b_\alpha^F \upharpoonright \delta = b_\bar{\alpha}^F \upharpoonright \delta \)
- If \( \bar{\alpha} < \alpha \) and \( \delta \in \text{dom}(b_\bar{\alpha}^F) \), then \( \delta \in \text{dom}(b_\alpha^F) \) and \( b_\alpha^F(\delta) = b_\bar{\alpha}^F(\delta) \).

Also, if \( \Delta \) is a set of model markers, we will say that \( \Delta \) is \textit{closed under copying via} \( \tau \) if for every sequence \( \mathcal{E} \) of edges coming from \( \tau \), every \( (M, \rho) \in \Delta \) such that \( M \in \text{dom}(\Psi_\mathcal{E}) \), and every \( \xi \in \mathcal{C}_\mathcal{E} \cap \text{cl}(M) \cap (\rho + 1) \) there is some \( \rho' \geq \Psi_\mathcal{E}(\xi) \) such that \( (\Psi_\mathcal{E}(M), \rho') \in \Delta \).

2.1. \textbf{Defining} \( \mathcal{P} \). Our forcing \( \mathcal{P} \) will be \( \mathcal{P}_{\omega_2} \), where \( (\mathcal{P}_\beta : \beta \leq \omega_2) \) is the sequence of posets to be defined next. Given \( \alpha \leq \omega_2 \), \( \dot{G}_\alpha \) will be the canonical \( \mathcal{P}_\alpha \)-name for the generic filter added by \( \mathcal{P}_\alpha \). We shall also denote the forcing relation for \( \mathcal{P}_\alpha \) by \( \Vdash_{\alpha} \), and the extension relation for \( \mathcal{P}_\alpha \) by \( \leq_{\alpha} \).

Given any \( \alpha \in \omega_2 \), and assuming \( \mathcal{P}_\alpha \) has been defined, we let \( \dot{C}^\alpha \) be some canonically chosen (using \( \Phi \)) \( \mathcal{P}_\alpha \)-name for a club–sequence on \( \omega_1^V \) such that \( \mathcal{P}_\alpha \) forces that

- \( \dot{C}^\alpha = \Phi(\alpha) \) in case \( \Phi(\alpha) \) is a \( \mathcal{P}_\alpha \)-name for a club–sequence on \( \omega_1 \), and that
• \(\mathcal{C}_\alpha\) is some fixed club–sequence on \(\omega_1\) in the other case.

Let \(\beta \leq \omega_2\) and suppose that \(\mathcal{P}_\alpha\) has been defined for every \(\alpha < \beta\). A triple \(q = (F_q, \Delta_q, \tau_q)\) is a \(\mathcal{P}_\beta\)-condition if and only if it has the following properties.

1. \(F_q\) is a relevant function such that \(\text{dom}(F_q) \subseteq \beta\).
2. \(\Delta_q\) is a countable set of models with markers such that
   (a) there are only finitely many \(N \in \text{dom}(\Delta_q)\) such that \((N, \rho) \in \Delta_q\) for some \(\rho > 0\), and
   (b) \(\text{dom}(\Delta_q)\) is a symmetric system.
3. \(\tau_q\) is a symmetric and reflexive relation on \(\Delta_q\).
4. There is a countable set \(\Delta\) of models with markers and a set \(\tau\) of edges from \(\Delta\) with the following properties.
   (a) \(\text{dom}(\Delta_q) = \text{dom}(\Delta)\), \(\{(N, \rho) \in \Delta_q : \rho > 0\} \subseteq \Delta\), and \(\tau_q \subseteq \tau\).
   (b) \(\tau\) is closed under copying and \(\Delta\) is closed under copying via \(\tau\).
   (c) \(\rho \leq \beta\) for every \((N, \rho) \in \Delta\).
   (d) For every sequence \(\mathcal{F} = (\langle (N^0_i, \rho^0_i), (N^1_i, \rho^1_i) \rangle \mid i < n)\) of edges from \(\tau\) and every \(\xi \in \mathcal{C}\) there is a sequence \(\mathcal{F}' = (\langle (N^0_i, \sigma^0_i), (N^1_i, \sigma^1_i) \rangle \mid i < n)\) of edges from \(\tau\) such that
      (i) for every \(i < n\), \(N^0_i \in Q^0_i\) and \(N^1_i = \Psi_{Q^0_i, Q^1_i}(N^0_i)\), and
      (ii) \(\xi \in \mathcal{C}\).
5. For every \(\alpha < \beta\), the restriction of \(q\) to \(\alpha\), \(q|_\alpha\), is a condition in \(\mathcal{P}_\alpha\), where
   \[q|_\alpha = (F_q|_\alpha, \Delta_q|_\alpha, \tau_q|_\alpha)\]
6. If \(\alpha \in \text{dom}(F_q)\), then \(F_q(\alpha) = (\mathcal{I}_\alpha^q, b^\alpha_q)\) has the following properties.
   (a) \(\{\min(I) : I \in \mathcal{I}_\alpha^q\} = \{\delta_N : N \in \mathcal{N}^q_{\alpha+1}\}\)
   (b) The following holds for every \(N \in \mathcal{N}^q_{\alpha+1}\) such that \(\delta_N \in \text{dom}(b^\alpha_q)\).
      (i) For every \(I \in \mathcal{I}_\alpha^q\) such that \(b^\alpha_q(\delta_N) < \min(I) < \delta_N\), \(q|_\alpha \Vdash \min(I) \notin \mathcal{C}^\alpha_{\delta_N}\).
      (ii) \(q|_\alpha\) forces that for every \(a \in N\) there is some \(M \in \mathcal{N}^q_{\alpha+1}\) such that \(\delta_M < \delta_N\), \(\delta_M \notin \mathcal{C}^\alpha_{\delta_N}\), and \(a \in \text{range}(\Psi_{\mathcal{M}, N})\).
7. \(F_q\) is closed under copying via \(\tau_q\).

\(\text{Few new reals}^{11}\) Strictly speaking, the reflexivity of \(\tau_q\) (i.e., the fact that \(\langle (N, \rho), (N, \rho) \rangle \in \tau_o\) for every \((N, \rho) \in \Delta_q\) is not needed, and in fact we obtain an isomorphic forcing notion if we drop this requirement. However, we are imposing this condition for notational convenience in the proof of Lemma 3.9.
(8) $\Delta_q$ is closed under copying via $\tau_q$.

(9) $\tau_q$ is closed under copying.

Given $\mathcal{P}_\beta$-conditions $q_i$, for $i = 0, 1$, $q_1 \leq_\beta q_0$ if and only if the following holds.

(1) For every $(N, \rho) \in \Delta_{q_0}$ there is some $\rho' \geq \rho$ such that $(N, \rho') \in \Delta_{q_1}$.

(2) $\text{dom}(F_{q_0}) \subseteq \text{dom}(F_{q_1})$ and the following holds for every $\alpha \in \text{dom}(F_{q_0})$,

(a) For every $I \in \mathcal{I}_\alpha$ there is some $I' \in \mathcal{I}_\alpha$ such that $\min(I') = \min(I)$.\(^{12}\)

(b) $b_{q_0}^\alpha \subseteq b_{q_1}^\alpha$

It is now time to give some motivation for our construction. Our aim is to build a finite support iteration of length $\omega_2$, along which we attempt to add clubs for measuring club–sequences handed down to us by our book–keeping function $\Phi$.\(^{13}\) The measuring club at a given coordinate $\alpha < \omega_2$ will be the collection $D$ of the minima of the intervals $I \in \mathcal{I}_\alpha$ for $q \in G$ such that $\alpha \in \text{dom}(F_q)$ (where $G$ is the generic filter).

For such a stage $\alpha$ of the construction and given $\delta \in D$, where $D$ is as above, we may sometimes also make the promise that a tail of the intersection with $\delta$ of $D$ stay outside of $\dot{C}_\delta$, namely the member $\dot{C}_\alpha$ indexed by $\delta$. This is expressed by putting $\delta$ in the domain of $b^\alpha_{q_0}$. The symmetry of $\text{dom}(\Delta_q)$ is part of what is needed in the verification that Measuring holds in the end; more precisely, in the verification that, for such $\alpha$ and $\delta$, a tail of the intersection with $\delta$ of $D$ will be forced to get inside $C_\delta$ in case we have not been in a position to make a promise as above at $\delta$.\(^{14}\)

The preservation of CH—more specifically, the fact that our forcing does not add more than $\aleph_1$-many new reals—is ensured by an argument as the one we described in the introduction. The relevant symmetry on both working parts and side conditions is dictated by the presence of the edges in $\tau_q$, through clauses (7)–(9) in the definition of condition. The imposition of this very strong type of symmetry—even if only locally, i.e., only for the information living in the models forming the edges—make it problematic to build a forcing iteration, at least in the

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\(^{12}\)Note that $I'$ is unique.

\(^{13}\)A standard $\Delta$-standard argument using CH will show that the forcing is $\aleph_2$-c.c., so every relevant name for a club–sequence may be assumed to be in $H(\omega_2)$ and will be picked out cofinally often by our book–keeping function.

\(^{14}\)The fact that $\text{dom}(\Delta_q)$ is infinite, together with clause (4) in the definition of condition, is also needed.
Few new reals

sense that $\mathcal{P}_\alpha$ be a complete suborder of $\mathcal{P}_\beta$ for all stages $\alpha < \beta$; and even if were happy to renounce to the construction being an iteration in this sense, these strong symmetry constraints make it problematic to ultimately define the construction without circularity. These difficulties essentially boil down to the type of situation in which we have an edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau_q$ and a copying point $\alpha \in \mathcal{C}^{N_0, \rho_0}_{N_1, \rho_1}$ such that $\alpha$ and $\alpha' := \Psi_{N_0, N_1}(\alpha)$ are such that $\delta_{N_0} \in \text{dom}(b^q_\alpha)$ and $\delta_{N_1} = \delta_{N_0} \in \text{dom}(b^q_{\alpha'})$. There is a priori no reason why there should be any agreement between $\dot{C}^\alpha_{\delta_{N_0}}$ and $\dot{C}^{\alpha'}_{\delta_{N_1}}$, and this is the main source of problems we were alluding to. These problems are solved by requiring that in a situation like the above we have another edge subsuming $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$, in the sense specified by clause (6)(b)(c). Thanks to this manoeuvre, $\dot{C}^\alpha_{\delta_{N_0}}$ and $\dot{C}^{\alpha'}_{\delta_{N_1}}$ will actually be forced, in the above situation, to be identical.

3. Proving Theorem 1.2

In this section we will prove the lemmas that, together, will yield a proof of Theorem 1.2. Our first two lemmas are obvious.

**Lemma 3.1.** For all $\alpha < \beta \leq \omega_2$, $\mathcal{P}_\alpha \subseteq \mathcal{P}_\beta$.

**Lemma 3.2.** For every $\beta < \omega_2$, $\mathcal{P}_\beta$ is definable over the structure $(H(\omega_2); \in, T_{\beta+1})$ without parameters. Moreover, this definition can be taken to be uniform in $\beta$.

A partial order $\mathbb{P}$ is $\aleph_2$-Knaster if for every sequence $(q_\alpha : \alpha < \omega_2)$ of $\mathbb{P}$-conditions there is a set $I \subseteq \omega_2$ of cardinality $\aleph_2$ such that $q_\alpha$ and $q_{\alpha'}$ are compatible for all $\alpha, \alpha' \in I$. Every $\aleph_2$-Knaster partial order has of course the $\aleph_2$-chain condition.

Given relevant function $F_0$ and $F_1$, let $F_0 + F_1$ denote the function $F$ with domain $\text{dom}(F_0) \cup \text{dom}(F_1)$ such that for every $\alpha \in \text{dom}(F)$,

- $F(\alpha) = F_\epsilon(\alpha)$ if $\epsilon \in \{0, 1\}$ and $\alpha \in \text{dom}(F_\epsilon) \setminus \text{dom}(F_{1-\epsilon})$, and
- if $\alpha \in \text{dom}(F_0) \cap \text{dom}(F_1)$, then

  $F(\alpha) = (\mathcal{I}^F_0 \cup \mathcal{I}^F_1, b^F_0 \cup b^F_1)$

Our next lemma is proved by a standard $\Delta$-system argument using $\text{CH}$. We sketch the argument for completeness (see e.g. [5] or [8] for similar arguments).
Lemma 3.3. For every $\beta \leq \omega_2$, $\mathcal{P}_\beta$ is $\aleph_2$-Knaster.

Proof. Let $q_\nu$ ($\nu < \omega_2$) be a sequence of $\mathcal{P}_\beta$-conditions. Let $\tilde{T} = \{(\alpha, x) : \alpha < \omega_2, x \in T_\alpha\}$. For each $\nu \in \omega_2$, we find an auxiliary countable $N^\nu < (H(\omega_2); \in, \tilde{T})$ such that $q_\nu \in N^\nu$. By CH there is $X \in [\omega_2]^{\aleph_2}$ and a set $R$ such that

$$N^\nu \cap N^{\nu'} = R$$

for all distinct $\nu, \nu' \in X$. Using again CH, by shrinking $X$ if necessary we may assume that, for some $m < \omega$, there are, for all $\nu \in X$, enumerations $((N^\nu_i, \rho^\nu_i) : i < \omega), (((N^{\nu_0}_0, \rho^{\nu_0}_0), (N^{\nu_1}_1, \rho^{\nu_1}_1)) : i < \omega)$ and $((\xi^\nu_i : j < m))$ of $\Delta_{q_\nu}, \tau_{q_\nu}$ and dom($F_{q_\nu}$), respectively, such that for all $\nu_0 \neq \nu_1$ in $X$ there is an isomorphism $\Psi$ between the structures

$$\langle N^{\nu_0}; \in, R, N_i^{\nu_0}, \rho_i^{\nu_0}, N^0_{\nu_1}, \rho_0^0_{\nu_1}, N^1_{\nu_1}, \rho^1_{\nu_1}, \xi^0_i, \xi^1_i, \tilde{T}_{\nu_0}, \tilde{T}_{\nu_1} \rangle_{i<\omega,j<m}$$

and

$$\langle N^{\nu_1}; \in, R, N_i^{\nu_1}, \rho_i^{\nu_1}, N^0_{\nu_1}, \rho_0^0_{\nu_1}, N^1_{\nu_1}, \rho^1_{\nu_1}, \xi^0_i, \xi^1_i, \tilde{T}_{\nu_1} \rangle_{i<\omega,j<m}.$$  

Note that $\Psi$ is the identity on $R$ by Fact 2.3. It is then immediate to verify, using Lemmas 2.4 and 2.5, that for all $\nu_0 \neq \nu_1$ in $X$,

$$(F_{q_{\nu_0}} + F_{q_{\nu_1}}, \Delta_{q_{\nu_0}} \cup \Delta_{q_{\nu_1}}, \tau_{q_{\nu_0}} \cup \tau_{q_{\nu_1}})$$

is a condition in $\mathcal{P}_\beta$ extending both $q_{\nu_0}$ and $q_{\nu_1}$. \hfill $\Box$

The following lemma, which will be used repeatedly, follows immediately from Lemma 3.2.

Lemma 3.4. Let $\mathcal{E}$ be a sequence of edges and suppose $\alpha + 1 \in \mathcal{C}^\mathcal{E}$, $a \in \text{dom}(\Psi_{\mathcal{E}})$, and $r \in \mathcal{P}_\alpha \cap \text{dom}(\Psi_{\mathcal{E}})$ is a condition forcing, in $\mathcal{P}_\alpha$, that

$$(H(\omega_2); \in, \tilde{T}_{\alpha+1}) \models \varphi(a),$$

for some formula $\varphi(x)$ in the language for $(H(\omega_2); \in, \tilde{T}_{\alpha+1})$. Let $\tilde{\alpha} = \Psi_{\mathcal{E}}(\alpha)$. Then $\Psi_{\mathcal{E}}(r)$ is a condition forcing, in $\mathcal{P}_\alpha$, that

$$(H(\omega_2); \in, \tilde{T}_{\alpha+1}) \models \varphi(\Psi_{\mathcal{E}}(a))$$

Given sets $\tau^0$ and $\tau^1$ of edges, we will denote by $\tau^0 \oplus \tau^1$ the natural closure of $\tau^0 \cup \tau^1$ under relevant isomorphisms $\Psi_{\mathcal{N}_0, \mathcal{N}_1}$ so as to obtain a set of edges closed under copying. More specifically, we define $\tau^0 \oplus \tau^1$ as $\bigcup_{i<\omega} \tau_i$, where $(\tau_i)_{i<\omega}$ is the following sequence.

- $\tau_0 = \tau^0 \cup \tau^1$
• For every $n < \omega$, $\tau_{n+1}$ is the union of $\tau_n$ and the set of edges of the form

$$\langle (\Psi_\mathcal{E}(N_0), \Psi_\mathcal{E}(\xi_0)), (\Psi_\mathcal{E}(N_1), \Psi_\mathcal{E}(\xi_1)) \rangle,$$

where $\mathcal{E}$ is a sequence of edges from $\tau_n$, $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau_n \cap \text{dom}(\Psi_\mathcal{E})$, $\xi_0 \in C^\mathcal{E} \cap N_0 \cap (\rho_0 + 1)$, and $\xi_1 \in C^\mathcal{E} \cap N_0 \cap (\rho_1 + 1)$.

Given $\tau_0$ and $\tau_1$ as above, the construction of $\tau_0 \oplus \tau_1$ as $\bigcup_{n<\omega} \tau_n$ gives rise to a natural notion of rank on $\tau_0 \oplus \tau_1$. Specifically, given $n < \omega$, $e = \langle (N_0, \gamma_0), (N_1, \gamma_1) \rangle \in \tau_0 \oplus \tau_1$ has $(\tau_0, \tau_1)$-rank $n$ iff $n$ is least such that $e$ comes from $\tau_n$. Alternatively, we may define the $(\tau_0, \tau_1)$-rank of $e$ as follows.

• $e$ has $(\tau_0, \tau_1)$-rank 0 if $e \in \tau_0 \cup \tau_1$.
• For every $n < \omega$, $e \in \tau_0 \oplus \tau_1$ has $(\tau_0, \tau_1)$-rank $n + 1$ iff $e$ does not have $(\tau_0, \tau_1)$-rank $m$ for any $m \leq n$ and there is a sequence $\mathcal{E} = \langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) : i \leq l \rangle$ of edges in $\tau_0 \oplus \tau_1$, along with an edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in (\tau_0 \oplus \tau_1) \cap \text{dom}(\Psi_\mathcal{E})$ such that
  - the maximum of the $(\tau_0, \tau_1)$-ranks of $\langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) \rangle$ (for $i \leq l$) and $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ is $n$, and
  - there are $\xi_0 \in C^\mathcal{E} \cap N_0 \cap (\rho_0 + 1)$ and $\xi_1 \in C^\mathcal{E} \cap N_0 \cap (\rho_1 + 1)$ such that $e = \langle (\Psi_\mathcal{E}(N_0), \Psi_\mathcal{E}(\xi_0)), (\Psi_\mathcal{E}(N_1), \Psi_\mathcal{E}(\xi_1)) \rangle$.

Let us say that a set $\tau$ is symmetric if for every edge $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$, $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau$ if and only if $\langle (N_1, \rho_1), (N_0, \rho_0) \rangle \in \tau$.

**Remark 3.5.** If $\tau_0$ and $\tau_1$ are symmetric sets of edges, then $\tau_0 \oplus \tau_1$ is also symmetric and for every $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle \in \tau_0 \oplus \tau_1$ and $n < \omega$, $\langle (N_0, \rho_0), (N_1, \rho_1) \rangle$ and $\langle (N_1, \rho_1), (N_0, \rho_0) \rangle$ have the same $(\tau_0, \tau_1)$-rank.

Given a sequence $\mathcal{E} = \langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) : i < n \rangle$ of edges, let $\mathcal{E}^{-1}$ denote $\langle (N_0^{n-i}, \rho_0^{n-i}), (N_1^{n-i}, \rho_1^{n-i}) : i < n \rangle$.

It will be convenient to isolate the following lemma.

**Lemma 3.6.** For all symmetric sets $\tau_0$ and $\tau_1$ of edges, every set $x$, and every $\tau_0 \oplus \tau_1$-thread $\langle \mathcal{E}, x \rangle$ there is a $\tau_0 \cup \tau_1$-thread $\langle \mathcal{E}_*, x \rangle$ such that

$$\Psi_\mathcal{E}(x) = \Psi_{\mathcal{E}_*}(x)$$

Furthermore, if $x = (y, \alpha) \in H(\omega_2) \times \omega_2$ and $\langle \mathcal{E}, x \rangle$ is correct, then $\mathcal{E}_*$ may be chosen to be correct as well,

**Proof.** Let $\mathcal{E} = \langle e_i : i \leq n \rangle$, where $e_i = \langle (N_0^i, \rho_0^i), (N_1^i, \rho_1^i) \rangle$ for each $i$. We aim to prove that there is a $\tau_0 \oplus \tau_1$-thread $\langle \mathcal{E}_*, x \rangle$ with the following properties.

1. $\Psi_\mathcal{E}(x) = \Psi_{\mathcal{E}_*}(x)$
(2) If every \( e_i \) has \((\tau^0, \tau^1)\)-rank 0, then \( \mathcal{E}_s = \mathcal{E} \).

(3) If some \( e_i \) has positive \((\tau^0, \tau^1)\)-rank, then the maximum \((\tau^0, \tau^1)\)-rank of the members of \( \mathcal{E}_s \) is strictly less than the maximum \((\tau^0, \tau^1)\)-rank of the members of \( \mathcal{E} \).

(4) The following holds, where \( \mathcal{E}_s = (e^*_i \mid i \leq n^*) \).

(a) \( e^*_0 = e_0 \) if \( e_0 \in \tau^0 \cup \tau^1 \).

(b) If there is a sequence \( \mathcal{F} = (\langle (M_k^0, \sigma_0^k), (M_1^k, \sigma_1^k) \rangle : k \leq m) \) of edges in \( \tau^0 \cup \tau^1 \) of \((\tau^0, \tau^1)\)-rank less than \( e_0 \), together with an edge \( \langle (M_0, \sigma_0), (M_1, \sigma_1) \rangle \in (\tau^0 \cup \tau^1) \cap \text{dom}(\Psi) \) of rank less than \( e_0 \) such that

\[
e_0 = \langle (\Psi_{\mathcal{F}}(M_0), \xi_0), (\Psi_{\mathcal{F}}(M_1), \xi_1) \rangle
\]

for some \( \xi_0 \in C^F \cap M_0 \cap (\sigma_0 + 1) \) and \( \xi_1 \in C^G \cap (\sigma_1 + 1) \), then

\[
e^*_0 = \langle (N_0^m, \sigma_0^m), (N_1^m, \sigma_1^m) \rangle,
\]

where \( \langle (N_0^m, \sigma_0^m), (N_1^m, \sigma_1^m) \rangle \) is some edge as above.

(c) \( e^*_n = e_n \) if \( e_n \in \tau^0 \cup \tau^1 \).

(d) If there is a sequence \( \mathcal{F} = (\langle (M_k^0, \sigma_0^k), (M_1^k, \sigma_1^k) \rangle : k \leq m) \) of edges in \( \tau^0 \cup \tau^1 \) of \((\tau^0, \tau^1)\)-rank less than \( e_n \), together with an edge \( \langle (M_0, \sigma_0), (M_1, \sigma_1) \rangle \in (\tau^0 \cup \tau^1) \cap \text{dom}(\Psi) \) of rank less than \( e_n \) such that

\[
e_n = \langle (\Psi_{\mathcal{F}}(M_0), \xi_0), (\Psi_{\mathcal{F}}(M_1), \xi_1) \rangle
\]

for some \( \xi_0 \in C^F \cap M_0 \cap (\sigma_0 + 1) \) and \( \xi_1 \in C^G \cap (\sigma_1 + 1) \), then

\[
e^*_n = \langle (N_0^m, \sigma_0^m), (N_1^m, \sigma_1^m) \rangle,
\]

where \( \langle (N_0^m, \sigma_0^m), (N_1^m, \sigma_1^m) \rangle \) is some edge as above.

It easily follows from the fact that (1) and (4) holds for all threads, that if \( x = (y, \alpha) \in H(\omega_2) \times \omega_2 \) and \( \langle \mathcal{E}, x \rangle \) is correct, then so is \( \mathcal{E}_s \).

The proof of (1)–(4) will be by induction on \( n \). We may assume that there is some \( i < n \) such that \( e_i \notin \tau^0 \cup \tau^1 \). Then there is a sequence \( \mathcal{F} = (\langle (M_k^0, \sigma_0^k), (M_1^k, \sigma_1^k) \rangle : k \leq m) \) of edges in \( \tau^0 \cup \tau^1 \) of \((\tau^0, \tau^1)\)-rank less than \( e_i \), along with an edge \( e = \langle (M_0, \sigma_0), (M_1, \sigma_1) \rangle \in (\tau^0 \cup \tau^1) \cap \text{dom}(\Psi_{\mathcal{F}}) \), also of rank less than \( e_i \), such that

\[
e_i = \langle (\Psi_{\mathcal{F}}(M_0), \xi_0), (\Psi_{\mathcal{F}}(M_1), \xi_1) \rangle
\]

for some \( \xi_0 \in C^F \cap M_0 \cap (\sigma_0 + 1) \) and \( \xi_1 \in C^G \cap (\sigma_1 + 1) \). By induction hypothesis there is a \( \tau^0 \cup \tau^1 \)-thread \( \langle \mathcal{E}_0, x \rangle \), together with a \( \tau^0 \cup \tau^1 \)-thread of the form \( \langle \mathcal{E}_2, \Psi_{\mathcal{E}_i+1}(x) \rangle \), such that

\[
\Psi_{\mathcal{E}_0}(x) = \Psi_{\mathcal{E}_i}(x)
\]
and

\[ \Psi_{\mathcal{E}}(\Psi_{\mathcal{E}_{i+1}}(x)) = \Psi_{\mathcal{E}_{i+1}}(x), \]

and such that the relevant instances of (1)–(4) hold for \( \langle \mathcal{E}_0, x \rangle \) and \( \langle \mathcal{E}_2, \Psi_{\mathcal{E}_{i+1}}(x) \rangle \). Also, by the choice of \( e_i \), the thread \( \langle \mathcal{E}_i, \Psi_{\mathcal{E}_{i+1}}(x) \rangle \) satisfies the instances of (1), (3) and (4) corresponding to \( \langle (e_i), \Psi_{\mathcal{E}_{i+1}}(x) \rangle \), where \( (e_i) \) is the sequence whose only member is \( e_i \), and where \( \mathcal{E}_1 \) is the concatenation of \( \mathcal{E}_{i+1}^{-1} \langle e \rangle \), and \( \mathcal{F} \). But now we may take \( \mathcal{E}_* \) to be the concatenation of \( \mathcal{E}_0, \mathcal{E}_1, \) and \( \mathcal{E}_2 \). \( \square \)

Given a symmetric system \( \mathcal{N} \) and \( \mathcal{P} \)-conditions \( q_0 \) and \( q_1 \), there is a natural notion of amalgamation of \( \{(M, N) : M \in \mathcal{N}\}, \emptyset, \emptyset \), \( q_0, q_1 \), which we will denote by \( q_0 \oplus \mathcal{N} \), \( q_1 \mathcal{N} \), \( q_0 \oplus \mathcal{N} \), \( q_1 \) is the triple \((F_{q_0 q_1}, \Delta_{q_0 q_1}, N, \tau_{q_0 \oplus \tau_{q_1}}, \mathcal{N}) \), where \( F_{q_0 q_1} \) and \( \Delta_{q_0 q_1} \mathcal{N} \) are as follows.

1. \( \Delta_{q_0 q_1, \mathcal{N}} \) is the union of
   (a) \( \Delta_{q_0} \cup \Delta_{q_1} \cup \{(M, 0) : M \in \mathcal{N}\} \)
   (b) the set of models with markers of the form \( (\Psi_{\mathcal{E}}(N_0), \xi) \), for
   some sequence \( \mathcal{E} \) of edges in \( \tau_{q_0 \oplus \tau_{q_1}} \) and some model with
   marker \( (N, \rho) \) in

   \[ (\Delta_{q_0} \cup \Delta_{q_1} \cup \{(M, 0) : M \in \mathcal{N}\}) \cap \text{dom}(\Psi_{\mathcal{E}}), \]

   and where \( \xi = \Psi_{\mathcal{E}}(\xi) \) for some \( \xi \in C \mathcal{E} \cap \mathcal{N} \cap (\rho + 1) \).

2. \( F_{q_0 q_1} \) is the function with domain the set of ordinals of the form
   \( \Psi_{\mathcal{E}}(\alpha) \), where \( \alpha \in \text{dom}(F_{q_0}) \cup \text{dom}(F_{q_1}) \), \( \mathcal{E} \) is a sequence of edges in \( \tau_{q_0 \oplus \tau_{q_1}} \), and \( \alpha \in C \mathcal{E} \), and such that for every \( \alpha \in \text{dom}(F_{q_0 q_1}) \),
   \( F_{q_0 q_1}(\alpha) = (\mathcal{I}_{q_0 q_1}^a, b_{q_0 q_1}^a) \), where \( \mathcal{I}_{q_0 q_1}^a \) and \( b_{q_0 q_1}^a \) are as follows.
   (a) \( \mathcal{I}_{q_0 q_1}^a \) is the union of
      (i) \( \mathcal{I}_{q_0}^a \cup \mathcal{I}_{q_1}^a \),
      (ii) the union of the sets of the form

      \[ \{I \in \mathcal{I}_{q_0}^a : \min(I) < \delta\}, \]

      for \( \epsilon \in \{0, 1\} \) and for \( \delta < \omega_1 \) and \( \alpha \in \text{dom}(F_{q_0}) \)
      such that there is a sequence \( \mathcal{E} \) of edges in \( \tau_{q_0 \oplus \tau_{q_1}} \),
      involving models of height at least \( \delta \) and such that
      \( \alpha \in C \mathcal{E} \) and \( \Psi_{\mathcal{E}}(\alpha) = \alpha \), and
      (iii) the set of singletons of the form \( \{\delta_M\} \), for \( M \in \mathcal{N}_{q_0 q_1}^{\Delta_{q_0 q_1}} \mathcal{N} \)
      such that there is no interval \( I \) in either \( \mathcal{I}_{q_0}^a \cup \mathcal{I}_{q_1}^a \) or the set in (ii) such that \( \delta_M = \min(I) \).
   (b) \( b_{q_0 q_1}^a \) is the union of
      (i) \( b_{q_0}^a \cup b_{q_1}^a \),

(ii) the union of the sets of the form $b^0_\alpha \upharpoonright \delta$, for $\epsilon \in \{0, 1\}$ and for $\delta < \omega_1$ and $\bar{\alpha} \in \text{dom}(F_{q_\epsilon})$ such that there is a sequence $\mathcal{E}$ of edges in $\tau_{q_0} \oplus \tau_{q_1}$ involving models of height at least $\delta$ and such that $\bar{\alpha} \in C^\mathcal{E}$ and $\Psi_{\mathcal{E}}(\bar{\alpha}) = \alpha$, and

(iii) the union of the sets of the form $b^0_\alpha \upharpoonright \delta + 1 =$, for $\epsilon \in \{0, 1\}$ and for $\delta < \omega_1$ and $\bar{\alpha} \in \text{dom}(F_{q_\epsilon})$ such that there is a sequence $\mathcal{E}$ of edges in $\tau_{q_0} \oplus \tau_{q_1}$ involving models of height at least $\delta$ and such that $\bar{\alpha} \in C^\mathcal{E}$ and $\Psi_{\mathcal{E}}(\bar{\alpha}) = \alpha$.

Lemma 3.7 follows immediately from the definition of $\tau_{q_0} \oplus_N \tau_{q_1}$, $F_{q_0,q_1}$ and $\Delta_{q_0,q_1,N}$, together with Lemma 3.6.

Lemma 3.7. The following holds for all $\mathcal{P}$-conditions $q_0$ and $q_1$ and for every symmetric system $N$.

(1) $\tau_{q_0,q_1}$ is closed under copying.

(2) If $F_{q_0,q_1}$ is a relevant function, then

(a) $F_{q_0,q_1}$ is closed under copying via $\tau_{q_0,q_1}$ and

(b) for every $\alpha \in \text{dom}(F_{q_0,q_1})$,

$$\{\min(I) : I \in \mathcal{T}_{q_0,q_1}\} = \{\delta_M : M \in N^{\Delta_{q_0,q_1,N}}\}$$

(3) $\Delta_{q_0,q_1,N}$ is closed under copying via $\tau_{q_0,q_1}$.

3.1. Properness. Given a countable elementary substructure $N$ of $H(\omega_2)$ and a $\mathcal{P}_\beta$-condition $q$, for some $\beta \leq \omega_2$, we will say that $q$ is $(N, \mathcal{P}_\beta)$-generic if and only if $q$ forces $\check{G}_\beta \cap A \cap N \neq \emptyset$ for every maximal antichain $A$ of $\mathcal{P}_\beta$ such that $A \in N$. Note that this is more general than the standard notion of $(N, \mathcal{P})$-genericity, for a forcing notion $\mathcal{P}$, which applies only if $\mathcal{P} \in N$. Indeed, in our situation $\mathcal{P}_\beta$ is of course never a member of $N$ if $N \subseteq H(\omega_2)$.

The properness of $\mathcal{P}_\beta$, for all $\beta \leq \omega_2$, is an immediate consequence of Lemmas 3.8 and 3.9. The first of these lemmas will be used also in the proof of Lemmas 3.15.

Lemma 3.8. Let $\beta \leq \omega_2$, $q \in \mathcal{P}_\beta$, let $\beta_0$ and $\beta_1$ be ordinals in $\beta + 1$, and let $N_0$ and $N_1$ be such that $q \in N_0 \cap N_1$ and such that $N_0 = N^0_{\beta_0}$ and $N_1 = N^0_{\beta_1}$, where $(N^i_0)_{i < \omega}$ and $(N^i_1)_{i < \omega}$ are sequences of models such that for all $i < \omega$,

(1) $N^0_0 \in T_{\beta_0+1}$ and $N^1_1 \in T_{\beta_1+1}$.

(2) $(N^i_0, q, T_{\beta_0}) \cong (N^i_1, q, T_{\beta_1})$, and

(3) $\Psi_{N^0_0,N^1_1}(N^0_0) = N^1_1$. 
Then there is an extension \( q^* \in \mathcal{P}_\beta \) of \( q \) such that \((N_0, \beta_0), (N_1, \beta_1)\) \( \in \tau_{q^*} \).

**Proof.** This is straightforward. It suffices to set
\[
q^* = (F^*, \Delta^*, \tau_q \cup \{((N_0, \beta_0), (N_1, \beta_1))\}),
\]
where \( \text{dom}(F^*) = \text{dom}(F_q) \) and for each \( \alpha \in \text{dom}(F) \),
\[
F^*(\alpha) = (T_\alpha^0 \cup \{\delta_N\}), b_\alpha^0),
\]
and where
\[
\Delta^* = \Delta_q \cup \{(N_0, \beta_0), (N_1, \beta_1)\} \cup \bigcup_{0 < i < \omega} \{(N_i^0, 0), (N_i^1, 0)\}.
\]

The central lemma in this section is the following.

**Lemma 3.9.** The following holds for all \( \beta < \omega_2 \).

1. If \( q \in \mathcal{P}_\beta \) and \( N \in \mathcal{N}_\beta^\gamma \cap \mathcal{T}_{\beta+1} \), then \( q \) is \((N, \mathcal{P}_\beta)\)-generic.
2. Let \( \gamma \leq \omega_2, \beta < \gamma \), and suppose \( q \in \mathcal{P}_\gamma \) and \( r \in \mathcal{P}_\beta \) are such that \( r \leq_\beta q|_\beta \). Then \( q \) and \( r \) are compatible in \( \mathcal{P}_\gamma \).

**Proof.** The proof will be by induction on \( \beta \). We start with the proof of (1). Let \( A \in N \) be a maximal antichain of \( \mathcal{P}_\beta \), and suppose that \( q \) extends some condition \( r_0 \in A \). We want to show that \( r_0 \in N \), and for this it suffices to show that \( q \) is compatible with a condition in \( A \cap N \). The case \( \beta = 0 \) follows at once from Lemma 2.4, so we will assume \( \beta > 0 \). We may assume that there is some \( \alpha_0 \in \text{dom}(F_q) \cap N \) such that \( \delta_N \in \text{dom}(b_{\alpha_0}^q) \) as the proof otherwise is a simpler version of the proof in this case.

**Claim 3.10.** There is an extension \( q^* \in \mathcal{P}_\beta \) of \( q \) for which there is some \( M \in \text{dom}(\Delta_q^\gamma) \cap \mathcal{T}_1 \) of height less than \( N \) and some \( \eta < \delta_M \) such that the following holds.

1. \( A \in \text{range}(\Psi_{M, N}^{\text{dom}(\Delta_q^\gamma)}) \)
2. \( \delta_M \notin \text{dom}(b_{\alpha}^q) \) for any sequence of edges from \( \tau_{q^*} \) involving models of height at least \( \delta_M \) and any \( \alpha \in \text{dom}(F_{q^*}) \cap \text{dom}(\Psi_\xi) \) such that \( \Psi_\xi(\alpha) \in \text{range}(\Psi_{M, N}^{\text{dom}(\Delta_q^\gamma)}), \) and such that
3. \( q^*|_\alpha \models [\eta, \delta_M] \cap \check{C}_\alpha^\eta = \emptyset \) whenever \( \check{E} \) is a sequence of edges from \( \tau_{q^*} \) involving models of height at least \( \delta_M, \alpha \in \text{dom}(F_{\tau_0}) \cap \text{dom}(\Psi_\xi) \) is such that \( \Psi_\xi(\alpha) \in \text{range}(\Psi_{M, N}^{\text{dom}(\Delta_q^\gamma)}), \) \( \delta \in \text{dom}(b_{\alpha}^q) \) is above \( \delta_M \), and \( b_{\alpha}^q(\delta) < \delta_M \).
Proof. By extending $q$ if necessary, we may assume that there is a sequence $\mathcal{E}' = ((N_0^0, \rho_0^0), (N_i^1, \rho_i^1) : i \leq n)$ of edges from $\tau_q$ involving models of height greater than $\delta_N$ such that $\alpha_0 + 1 \in C_{\mathcal{E}'}$ and such that, letting $\tilde{\alpha}_0 = \Psi_{\mathcal{E}'}(\alpha_0)$, there is no sequence $\tilde{\mathcal{F}}$ of edges from $\tau_q$ involving models of height at least $\delta_N$ and such that $\tilde{\alpha}_0 + 1 \in C_{\tilde{\mathcal{F}}}$ and $\Psi_{\tilde{\mathcal{F}}}(\tilde{\alpha}_0) < \tilde{\alpha}_0$. Note that $\delta_N \in \text{dom}(b^\eta_{\alpha_0})$. Using an instance of clause (6)(b)(ii) in the definition of condition together with the openness of $C_\delta^\alpha$ in $V^{P_{\alpha_0}}$ and together with (2)$\tilde{\alpha}_0$, which is true by the induction hypothesis, we may find an extension $q_0 \in P_{\alpha_0}$ of $\hat{q}$ for which there is some $M_0 \in N_{\alpha_0}^{\alpha_0} \cap \mathcal{T}_{\alpha_0+1}$ and some $\eta_0 < \delta_M$ such that $\delta_Q < \delta_M$ for every $Q \in \text{dom}(\Delta_q)$ with $\delta_0 < \delta_N$, $\delta_M < \delta_N$, $A \in \text{range}(\Psi_{M_0,N}^{\text{dom}(\Delta_0)})$, and such that $q_0 \models [\eta_0, \delta_M] \cap C_{\alpha_0}^\eta = \emptyset$. Specifically, we obtain $q_0$ as the minimal amalgamation of $q$ and some $s \in \mathcal{P}_{\alpha_0}$ extending $q \upharpoonright \tilde{\alpha}_0$ for which there is some $M_0 \in N_{\alpha_0}^{\alpha_0} \cap \mathcal{T}_{\alpha_0+1}$ such that $A \in \text{range}(\Psi_{M_0,N}^{\text{dom}(\Delta_0)})$, $\delta_Q < \delta_M$ for every $Q \in \text{dom}(\Delta_q)$ with $\delta_Q < \delta_N$ and $\delta_M < \delta_N$, and such that $s \models [\eta_0, \delta_M] \cap C_{\alpha_0}^\eta = \emptyset$.

Subclaim 3.11. If $\alpha \in \text{dom}(F_{q_0}) \cap \text{range}(\Psi_{M_0,N}^{\text{dom}(\Delta_0)})$ and $\delta \in \text{dom}(b^\eta_\alpha)$ are such that $\alpha > \alpha_0$ and $\delta \geq \delta_M$, then in fact $\delta \in \text{dom}(b^\eta_\alpha)$.

Proof. Otherwise, by the way we have constructed $q_0$ from $q$ and $s$, there would be some sequence $\tilde{\mathcal{G}}$ of edges from $\tau_q$ involving models of height at least $\delta_N$ and such that $\alpha \in C_{\tilde{\mathcal{G}}}$ and $\Psi_{\tilde{\mathcal{G}}}(\alpha) < \tilde{\alpha}_0$. By the symmetry of $\text{dom}(\Delta_q)$ together with Fact 2.3, we would then have that $\alpha_0 + 1 \in C_{\tilde{\mathcal{G}}}$. But $\Psi_{\tilde{\mathcal{G}}}(\alpha_0) < \Psi_{\tilde{\mathcal{G}}}(\alpha) < \tilde{\alpha}_0$, and hence the concatenation $\tilde{\mathcal{F}}$ of $\tilde{\mathcal{E}}^{-1}$ and $\tilde{\mathcal{G}}$ would be a sequence of edges from $\tau_q$ involving models of height greater than $\delta_N$ and such that $\tilde{\alpha}_0 + 1 \in C_{\tilde{\mathcal{F}}}$ and $\Psi_{\tilde{\mathcal{F}}}(\tilde{\alpha}_0) < \tilde{\alpha}_0$, which is a contradiction. \hfill \Box

If there is some $\alpha_1 \in \text{dom}(F_{q_0}) \cap \text{range}(\Psi_{M_0,N}^{\text{dom}(\Delta_0)})$ such that $\delta_M \in \text{dom}(b^\eta_{\alpha_1})$, we may similarly find an extension $q_1 \in P_{\beta}$ of $q_0$ for which there is some $M_1 \in N_{\alpha_1}^{\alpha_1} \cap \mathcal{F}_{\alpha_1+1}$ and some $\eta_1 < \delta_{M_1}$ such that $\delta_{M_1} < \delta_M$, $A \in \text{range}(\Psi_{M_1,N}^{\text{dom}(\Delta_0)})$, $q_1 \models [\eta_1, \delta_{M_1}] \cap C_{\alpha_1}^{\eta_1} = \emptyset$, and such that if $\alpha \in \text{dom}(F_{q_1}) \cap \text{range}(\Psi_{M_1,N}^{\text{dom}(\Delta_q)})$ and $\delta \in \text{dom}(b^\eta_{\alpha})$ are such that $\alpha > \alpha_1$ and $\delta \geq \delta_{M_1}$, then in fact $\delta \in \text{dom}(b^\eta_{\alpha})$. Proceeding in this way, and since $\delta_N > \delta_M > \delta_{M_1} > \ldots$, we end up with an extension $q^* \in P_{\beta}$ of $q$ for which there is some $M \in \text{dom}(\Delta_{q^*}) \cap \mathcal{T}_1$ of height less than $N$ and some $\eta < \delta_M$ such that the conclusion of the claim holds for $q^*$, $M$ and $\eta$. \hfill \Box
Let now $q^*$, $M \in \text{dom}(\Delta_{q^*}) \cap T$ of height less than $N$, and $\eta < \delta_M$ be as in the conclusion of the claim. Let $M^* := \text{range}(\Psi_{M,N}^{\text{dom}(\Delta_{q^*})})$. Let $G$ be $\mathcal{P}_0$-generic and such that $q^* \upharpoonright 0 \in G$. Note that $\delta_{M[G]} = \delta_M$ by (1), which is true by induction hypothesis. Let $B \in M$ be such that $\Psi_{M,N}^{\text{dom}(\Delta_{q^*})}(B) = A$. Let $\xi_0, \ldots, \xi_{m-1}$ be the ordinals in $F_{r_0} \cap M^*$ and for each $i < m$, let $\xi_i \in M$ be such that $\Psi_{M,N}^{\text{dom}(\Delta_{q^*})}(\xi_i) = \xi_i^*$. Also, let $\xi_0^*, \ldots, \xi_{m-1}^*$ be the ordinals in $(\text{dom}(F_{q^*}) \cap M^*) \setminus \text{dom}(F_{r_0})$ and let $\xi_0, \ldots, \xi_{n-1} \in M$ be such that $\Psi_{M,N}^{\text{dom}(\Delta_{q^*})}(\xi_j) = \xi_j^*$ for all $j < n$. Working in $M[G]$, we may find some $\bar{M}, \bar{N} \in \mathcal{N}_0^{q^*}$, all of these objects in $M[G]$, such that $\eta < \delta_M < \delta_N$; $\xi_i \in \bar{M}$ for all $i < m$, $\xi_j \in \bar{M}$ for all $j < n$, and such that there is some $\bar{r} \in \text{range}(\Psi_{\bar{M},\bar{N}}^{N_0^{q^*}}(B) )$ such that the following holds, where $\bar{\xi}_i = \Psi_{\bar{M},\bar{N}}^{N_0^{q^*}}(\xi_i)$ for each $i < m$ and $\bar{\xi}_j = \Psi_{\bar{M},\bar{N}}^{N_0^{q^*}}(\xi_j)$ for each $j < n$.

1. $\text{dom}(\Delta_{\bar{r}}) \subseteq \mathcal{N}_0^{q^*}$
2. The following holds for every $i < m$.
   (a) $\bar{\xi}_i \in \text{dom}(F_{\bar{r}})$
   (b) $\mathcal{T}_{\bar{\xi}_i}^{\bar{r}} \cap M \subseteq \mathcal{T}_{\bar{\xi}_i}^{\bar{r}}$
   (c) For every $I \in \mathcal{T}_{\bar{\xi}_i}^{\bar{r}}$, if $\min(I) \notin \{\min(J) : J \in \mathcal{T}_{\bar{\xi}_i}^{\bar{r}}\} \cap \delta_M$, then $\min(I) > \eta$.
   (d) $b_{\bar{\xi}_i}^{\bar{r}} \upharpoonright \delta_M$ is an initial segment of $b_{\bar{\xi}_i}^{\bar{r}}$.
3. $\{\bar{\xi}_0, \ldots, \bar{\xi}_{m-1}\} \cap \text{dom}(F_{\bar{r}}) = \emptyset$
4. For every $j < n$, $\{\delta_Q : Q \in \mathcal{N}_0^{\bar{\xi}_j+1}\} \cap \delta_M$ is an initial segment of $\{\delta_Q : Q \in \mathcal{N}_0^{\bar{\xi}_j+1}\}$, and every ordinal in $\{\delta_Q : Q \in \mathcal{N}_0^{\bar{\xi}_j+1}\}$ which is not in $\{\delta_Q : Q \in \mathcal{N}_0^{\bar{\xi}_j+1}\} \cap \delta_M$ is above $\eta$.

Indeed, the existence of $\bar{M}$ and $\bar{N}$ for which there is some $\bar{r} \in \text{range}(\Psi_{\bar{M},\bar{N}}^{N_0^{q^*}}(B) )$ as above is witnessed by $M$ and $N$, respectively, together with $r_0$; and, furthermore, the existence of such $\bar{M}$ and $\bar{N}$ may be expressed over $(H(\omega_2)[G]; \in, G)$ by a sentence with $B$, along with other objects in $M[G]$, as parameters. Still working in $M[G]$, let $Q \in \mathcal{N}_0^{\bar{g}} \cap M[G]$ be such that $\bar{r} \in Q$. Then $r := \Psi_{\bar{Q},\bar{M}}^{N_0^{q^*}}(B) \cap \bar{M}$, and therefore also $r^* := \Psi_{\bar{M},\bar{N}}^{\text{dom}(\Delta_{q^*})}(r) \in A \cap N$. Notice that we also have the following.

1. $\text{dom}(\Delta_{q^*}) \subseteq \mathcal{N}_0^{q^*}$
2. The following holds for every $i < m$.
   (a) $\xi_i^* \in \text{dom}(F_{q^*})$
   (b) $\mathcal{T}_{\xi_i}^{q^*} \cap M \subseteq \mathcal{T}_{\xi_i}^{q^*}$
(c) For every $I \in \mathcal{I}_\zeta^*$, if $\min(I) \notin \{\min(J) : J \in \mathcal{I}_\zeta^* \} \cap \delta_M$, then $\min(I) > \eta$.

(d) $b_{\zeta_i}^n \upharpoonright \delta_M$ is an initial segment of $b_\zeta^*$.

(3) $\{\zeta_0, \ldots, \zeta_{n-1}\} \cap \text{dom}(F_r) = \emptyset$

(4) For every $j < n$, $\{\delta_Q : Q \in \mathcal{N}_{\zeta_j+1}^\alpha\} \cap \delta_M$ is an initial segment of $\{\delta_Q : Q \in \mathcal{N}_{\zeta_j+1}^\alpha\}$, and every ordinal in $\{\delta_Q : Q \in \mathcal{N}_{\zeta_j+1}^\alpha\}$ which is not in $\{\delta_Q : Q \in \mathcal{N}_{\zeta_j+1}^\alpha\} \cap \delta_M$ is above $\eta$.

Let $q^{**} = q^* \oplus_N r^*$, where $N = \mathcal{A}(\text{dom}(\Delta_{q^*}), \text{dom}(\Delta_{r^*}), M^*)$. It follows from (1)–(4) together our choice of $q^*$, $M$ and $\eta$, and Lemmas 2.4 and 3.7, that $q^{**}$ is a condition in $\mathcal{P}_\beta$. This finishes the proof of (1)$_\beta$ since then $q^{**}$ is a common extension of $q^*$ and $r^*$.

We are left with the proof of (2)$_\beta$. Using the induction hypothesis for all relevant $\alpha < \beta$, we may extend $r$ to a condition $r^* \in \mathcal{P}_\beta$ such that for every $\alpha \in \text{dom}(F_r)$, every $N \in \mathcal{N}_{\alpha+1}^\alpha$ such that $\delta_N \in \text{dom}(b_\zeta^*)$ and every $I \in \mathcal{I}_\alpha^*$ such that $b_\alpha^*(\delta_N) < \min(I) < \delta_N$ there is a condition $t \in \mathcal{P}_\alpha$ such that

1. $t \Vdash_\alpha \min(I) \notin \dot{\mathcal{C}}_\alpha^\beta$,
2. $r^|_\alpha \leq \alpha$, and 
3. $r^* \in N'$ for every $N' \in \mathcal{N}_{\alpha+1}^\alpha$ such that $\delta_{N'} > \delta_N$.

But now it is easy to prove, using Lemmas 3.4 and 3.7, that $q^* = q \oplus_N r^* \in \mathcal{P}_\gamma$, where $N = \text{dom}(\Delta_{r^*})$. This finishes the proof since then $q^*$ is a common extension of $r^*$ and $q$.

The following two corollaries follows from Lemmas 3.8 and 3.9 (and Lemma 3.3 for $\beta = \omega_2$).

**Corollary 3.12.** For all $\beta < \gamma \leq \omega_2$, $\mathcal{P}_\beta$ is a complete suborder of $\mathcal{P}_\gamma$.

**Corollary 3.13.** For every $\beta \leq \omega_2$, $\mathcal{P}_\beta$ is proper.

### 3.2. New reals

The forcing for adding $\aleph_1$-many mutually generic Cohen reals, denoted by $\text{Add}(\omega, \omega_1)$, is the collection of finite functions $p \subseteq \omega_1 \times 2$, ordered by reverse inclusion.

There are many ways to see that forcing with $\mathcal{P}$ adds at least $\aleph_1$-many new reals. The proof of the following lemma is one of them.

**Lemma 3.14.** $\mathcal{P}$ adds an $\text{Add}(\omega, \omega_1)$-generic filter over $V$.

**Proof.** Let $(\alpha_\nu)_{\nu < \omega_2}$ be the strictly increasing enumeration of the set of ordinals $\alpha$ such that $\Phi(\alpha)$ is a $\mathcal{P}_\alpha$-name for the constantly-$\emptyset$ club sequence on $\text{Lim}(\omega_1)$. Let $H$ be a $\mathcal{P}$-name for the set of $p \in \text{Add}(\omega, \omega_1)$ such that for all $\nu \in \text{dom}(p)$, $p(\nu) = 1$ if and only if, letting $\delta$ be the least height of a model in $\mathcal{N}_{\alpha_\nu+1}^{\mathcal{M}}$ for some $q \in \dot{G}_{\omega_2}$ such that $\alpha_\nu \in$
dom($F_q$) and $\delta \in b_{\alpha_1}^\beta$, $b_{\alpha_1}^\beta(\delta)$ is of the form $\delta + n$, where $\delta \in \text{Lim}(\omega_1)$ and $n < \omega$ is even. Is is then straightforward to see that $H$ is forced to be an $\text{Add}(\omega, \omega_1)$-generic filter over $V$. \hfill \Box

We will now put clauses (7)–(9) in the definition of condition to work. The following lemma is a counterpoint to Lemma 3.14. It shows that $\mathcal{P}$ does not add more than $\aleph_1$-many new reals, and hence this forcing preserves $\text{CH}$ (cf. the proof of Proposition 2.7 in [7] or the proof sketched in the introduction).

**Lemma 3.15.** (Few new reals) $\mathcal{P}$ adds not more than $\aleph_1$-many new reals.

**Proof.** Suppose, towards a contradiction, that there is a $\mathcal{P}$-condition $q$ and a sequence $(\dot{r}_\nu)_{\nu < \omega_2}$ of $\mathcal{P}$-names for subsets of $\omega$ such that $q \Vdash_{\omega_2} \dot{r}_\nu \neq \dot{r}_{\nu'}$ for all $\nu \neq \nu'$. Let $\beta < \omega_2$ such that $q \in \mathcal{P}_\beta$. By Lemma 3.3, for each $\nu < \omega_2$ we may assume that $\dot{r}_\nu \in H(\omega_2)$ and we may find $\beta_\nu < \omega_2$ above $\beta$ and such that $\dot{r}_\nu$ is in fact a $\mathcal{P}_{\beta_\nu}$ for a subset of $\omega$. We may assume that $\beta_\nu < \beta_\mu$ for $\nu < \nu_1$.

Let $\chi$ be a large enough cardinal. For each $\nu < \omega_2$ and for $i < \omega$, let $N_{\nu,i}^*$ be a countable elementary substructure of $H(\chi)$ containing $q$, $\Phi$, $\dot{r}_\nu$, and $\beta_\nu$, and such that $N_{\nu,i}^* \in N_{\nu,i+1}^*$, and let $N_{\nu,i} = N_{\nu,i}^* \cap H(\omega_2)$. Let also $\vec{T}$ be as in the proof of Lemma 3.3, i.e., $\vec{T} = \{(\alpha, x) : \alpha < \omega_2, x \in T_\alpha\}$.

Using $\text{CH}$ we may find indices $\nu_1 < \nu_0$ such that

$$(\bigcup_{i < \omega} N_{\nu_0,i}^*; \dot{r}_{\nu_0}, N_{\nu_0,i}^{T_\nu}; \vec{T})_{i < \omega}$$

and

$$(\bigcup_{i < \omega} N_{\nu_1,i}^*; \dot{r}_{\nu_1}, N_{\nu_1,i}^{T_\nu}; \vec{T})_{i < \omega}$$

are isomorphic structures. Let $N_0 = N_{\nu_0,0}$ and $N_1 = N_{\nu_1,0}$. By Lemma 3.8 we may find an extension $q^*$ of $q$ in $\mathcal{P}_{\beta_1}$, with $((N_0, \beta_{\nu_0}), (N_1, \beta_{\nu_1})) \in \tau_{q^*}$.

Let now $q'$ be any extension of $q^*$ and suppose, towards a contradiction, that $q' \Vdash_{\beta_{\nu_1}} n \in \dot{r}_{\nu_0} \Delta \dot{r}_{\nu_1}$ for some $n < \omega$. Let us assume that $q' \Vdash_{\beta_{\nu_1}} n \in \dot{r}_{\nu_0} \setminus \dot{r}_{\nu_1}$ (the proof in the case that $q^* \Vdash_{\beta_{\nu_1}} n \in \dot{r}_{\nu_1} \setminus \dot{r}_{\nu_0}$ is symmetrical to the proof in the present case).

By Lemma 3.9, $q^*$ is $(N_{\nu_0}^*, \mathcal{P}_{\beta_{\nu_0}})$-generic. Hence, by extending $q'$ if necessary we may assume that $q'$ extends a condition $r \in N_{\nu_0} \cap \mathcal{P}_{\beta_{\nu_0}}$ such that $r \Vdash_{\beta_{\nu_0}} n \in \dot{r}_{\nu_0}$. By clauses (7)–(9) in the definition of condition applied to $q'$ it follows that $q'$ extends $\Psi_{N_{\nu_0}, N_{\nu_1}}(r)$. But $\Psi_{N_{\nu_0}, N_{\nu_1}}(r) \Vdash_{\beta_{\nu_1}} n \in \Psi_{N_{\nu_0}, N_{\nu_1}}(\dot{r}_{\nu_0})$ by Lemmas 3.2 and 3.4, which yields a contradiction since $\Psi_{N_{\nu_0}, N_{\nu_1}}(\dot{r}_{\nu_0}) = \dot{r}_{\nu_1}$. \hfill \Box
3.3. Measuring. The following lemma completes the proof of Theorem 1.2.

Lemma 3.16. \( \mathcal{P} \) forces Measuring.

Proof. Let \( G \) be \( \mathcal{P} \)-generic and let \( \tilde{C} = (C_\delta : \delta \in \text{Lim}(\omega_1)) \in V[G] \) be a club–sequence on \( \omega_1 \). We want to see that there is a club of \( \omega_1 \) in \( V[G] \) measuring \( \tilde{C} \). By the \( \aleph_2 \)-c.c. of \( \mathcal{P} \) together with \( 2^{\aleph_1} = \aleph_2 \), we may assume that \( \tilde{C} = \tilde{C}_G \) for some \( \mathcal{P} \)-name \( \tilde{C} \in H(\omega_2) \) for a club–sequence on \( \omega_1 \). Again by the \( \aleph_2 \)-c.c. of \( \mathcal{P} \) and since \( \Phi^{-1}(\tilde{C}) \) is unbounded in \( \omega_2 \), by the choice of \( \Phi \), we may fix some \( \alpha < \omega_2 \) such that \( \tilde{C} \) is a \( \mathcal{P}_\alpha \)-name and \( \Phi(\alpha) = \tilde{C} \). We then have that \( \tilde{C}^\alpha = \Phi(\alpha) \).

We will show that

\[
D = \{\min(I) : I \in I_\alpha^q \text{ for some } q \in G \text{ with } \alpha \in \text{dom}(F_q)\}
\]

is a club of \( \omega_1 \) measuring \( \tilde{C} \). Let \( \hat{D} \) be the canonical name for \( D \). To start with, it is straightforward to see that \( \hat{D} \) is forced to be unbounded in \( \omega_1 \) (by Lemma 3.8 we can always extend any given \( q \in \mathcal{P}_{\alpha+1} \) to an \((N, \mathcal{P}_{\alpha+1})\)-pre-generic condition for any \( N \in \mathcal{T}_{\alpha+1} \) such that \( q \in N \), and every such condition forces that \( N \in D \)). To see that \( \hat{D} \) is also forced to be closed, suppose towards a contradiction that \( q \in \mathcal{P}_{\alpha+1} \) forces some ordinal \( \delta < \omega_1 \) to be a limit point of \( \hat{D} \) but there is no \( N \in \mathcal{N}_{\alpha+1} \) with \( \delta_N = \delta \). We may of course assume, without loss of generality, that there is some model \( N \in \mathcal{N}_{\alpha+1} \) such that \( \delta_N < \delta \). Let \( N \) be of maximal height among such models and let \( I \) be the unique interval in \( I_\alpha^q \) such that \( \min(I) = \delta_N \). Now we may extend \( q \) to a condition \( q' \) obtained by replacing \( I \) in \( I_\alpha^q \) with \( [\delta_N, \delta] \) (which is possible since necessarily \( \max(I) < \delta \) by our assumption that \( q \) forces \( \delta \) to be a limit point of \( \hat{D} \) and closing under appropriate isomorphisms \( \Psi_{N_0, N} \), so that clause (7) in the definition of condition holds for \( q' \). But that yields a contradiction since then \( q' \) forces that \( \hat{D} \cap \delta \) is bounded by \( \min(I) \).

For every \( \delta \in D \), if there is some \( q \in G \) such that \( \alpha \in \text{dom}(F_q) \) and \( \delta \in \text{dom}(b^\alpha_q) \), then a tail of \( D \cap \delta \) is disjoint from \( C_\delta \) (by a suitable instance of clause (6)(b)(i) in the definition of condition).

In order to finish the proof of the lemma it suffices to prove that if \( \delta \in D \) is a limit point of \( D \) and \( \delta \not\in \text{dom}(b^\alpha_q) \) for any \( q \in G \) such that \( \alpha \in \text{dom}(F_q) \), then a tail of \( D \cap \beta \) is contained in \( C_\delta \). So let \( q \) be a condition with \( \alpha \in \text{dom}(F_q) \) and \( N \in \mathcal{N}_{\alpha+1} \) such that \( \delta_N = \delta \), and such that \( \delta \not\in \text{dom}(b^\alpha_q) \) for any \( q' \in \mathcal{P} \) extending \( q \). We may of course assume that \( q \) forces that \( \delta \) is a limit point of \( \hat{D} \). It suffices to find an extension \( q^* \) of \( q \) and some \( \overline{\delta} < \delta \) with the property that if \( q' \in \mathcal{P}_{\alpha+1} \) extends \( q^* \) and \( M \in \mathcal{N}_{\alpha+1} \) is such that \( \overline{\delta} < \delta_M < \delta \), then \( q' \vDash \delta_M \in C_\delta^\alpha \).
By our choice of \( q \), it is easy to see that there is some \( a \in N \) such that \( q|_a \) forces that if \( M \in \mathcal{N}_a^{\alpha} \cap \mathcal{T}_{\alpha+1} \) is such that \( \delta_M \prec \delta_N \), \( a \in \text{range}(\Psi_N^{\mathcal{N}_a}) \), then \( \delta_M \in \mathcal{C}_a^{\alpha} \). Indeed, if that were not the case, then thanks to clause (4) for \( q \) we would be able to extend \( q \) to a condition \( q' \) such that \( \delta \in \text{dom}(\mathfrak{b}^{\mathcal{A}}_a) \) (i.e., we would be able to do so while, if necessary, adding suitable edges ensuring that the relevant instance of clause (6)(b)(iii) in the definition of condition holds for \( q' \)). Now, thanks to conclusion (1) in Lemma 3.9, we can extend \( q \) to a condition \( q^* \) with a model \( Q \in \text{dom}(D_{q^*}) \cap N \) such that \( a \in Q \). Let \( \delta = \delta_Q \).

We now show that \( q^* \) and \( \delta \) are as desired. For this, suppose \( q^* \in \mathcal{P}_{\alpha+1} \) extends \( q^* \) and \( M \in \mathcal{N}_{\alpha+1}^q \) is such that \( \bar{\delta} < \delta_M < \delta \). In order to finish the proof, it suffices to show that \( q^* \models \alpha \delta \in \mathcal{C}_a^{\alpha} \). But by symmetry of \( \text{dom}(\Delta_{q^*}) \), \( Q \in \text{range}(\Psi_{\mathcal{N}_a}^{\text{dom}(\Delta_{q^*})}) \) and therefore also \( a \in \text{range}(\Psi_{\mathcal{N}_a}^{\text{dom}(\Delta_{q^*})}) \). It follows that \( q^* \models \alpha \delta \in \mathcal{C}_a^{\alpha} \) by our assumption on \( q \), which concludes the proof. \( \square \)

4. SOME CONCLUDING REMARKS

It will be sensible to finish the paper with some words addressing the issue of what goes wrong if we try to modify the present forcing so as to force \( \text{Unif}(\mathcal{C}) \), for some given ladder system \( \mathcal{C} = (C_\delta : \delta \in \text{Lim}(\omega_1)) \), together with \( \text{CH} \) as we mentioned in the introduction, the conjunction of these two statements cannot hold. One could in fact try to build something like a sequence of partial orders \( (\mathcal{P}_\alpha)_{\alpha \leq \omega_2} \) in our construction in such a way that, at every step \( \alpha < \omega_2 \), we attempt to add a uniformizing function on \( \mathcal{C}_\delta \) for some colouring \( F : \text{Lim}(\omega_1) \to \{0,1\} \) fed to us by our book-keeping function \( \Phi \). Thus, rather than the present triples \( (f,b,c) \), we would plug in conditions for a natural forcing for adding such a uniformizing function with finite conditions.

Everything would seem to go well, and in particular our construction would be a forcing iteration with the \( \aleph_2 \)-c.c., would be proper, and would preserve \( \text{CH} \), except that, because of the copying constraint expressed in the corresponding version of clauses (7)–(9) in the definition of condition, it would not be able to force \( \text{Unif}(\mathcal{C}) \). The reason is that we would not be in a position to rule out situations in which there is a condition \( q \) with, for example, an edge \( \langle (N_0, \rho_0), (N_1, \rho_1) \rangle \) in \( \tau_q \) for which there is some \( \alpha \in \mathcal{C}_p^{N_0,\rho_0} \) such that the colour of \( \hat{F}(\alpha) \) at \( \delta_N \) is forced to be 0, whereas the colour of \( \hat{F}(\check{\alpha}) \), for \( \check{\alpha} = \Psi_{N_0,\rho_1}(\alpha) \), at \( \delta_N \) is forced to be 1 (where \( \hat{F}(\xi) \) denotes of course the name for the colouring to be uniformized at stage \( \xi \) of the construction). The requirement,
imposed by the current form of clauses (7) –(9), that any amount of information on the generic uniformizing function at the coordinate \( \alpha \) be copied over to the coordinate \( \tilde{\alpha} \) would then make it impossible for these generic uniformizing functions to be defined on any tail of \( C_\delta \).

Such an obstacle could be resolved by requiring the presence of an edge in \( \tau_\eta \) subsuming \( (N_0, \rho_0), (N_1, \rho_1) \) in the sense of clause (6)(b)(iii), but this would readily give rise to the presence of an infinite towers of edges and then we would need to account for the points corresponding to the limit of the heights of the models in this tower, so we would be faced again with the same problem. This type of problems does not arise when forcing Measuring due to the more lenient nature of the ‘guessing’ in this case; if we cannot get the club to eventually stay outside a given \( C_\delta \), then it has to eventually get inside, but the fact whether one of the other is the case is determined by the specific club-sequence being measured (and by the ‘shape’ of the surrounding condition, of course).

It may also be worth pointing out that the type of situation described above is a source of serious obstacles towards trying to force some reasonable forcing axiom to hold together with \( CH \) using the present methods. To see this in a particularly simple case, suppose, for example, that \( (Q_\alpha)_{\alpha < \omega_2} \) is exactly as our present construction \( (P_\alpha)_{\alpha < \omega_2} \), except that at each stage we force with Cohen forcing. This construction enjoys all relevant nice properties that \( (P_\alpha)_{\alpha < \omega_2} \) has. On the other hand, \( Q_{\omega_2} \) obviously cannot possibly force \( FA_{(\forall \xi_1)}(Cohen) \), as it preserves \( CH \). Letting \( \alpha_0 < \omega_2 \) be such that all reals in \( V^{Q_{\omega_0}} \) have already appeared in \( V^{Q_{\omega_2}} \), if \( \alpha < \omega_2 \) is above \( \alpha_0 \), then the real constructed by the generic at the coordinate \( \alpha \) will actually fail to be Cohen–generic over \( V^{P_{\alpha_0}} \); in fact, for every condition \( q \in P_{\omega_2} \) such that \( \alpha \in \text{dom}(F_q) \) there will be a condition \( q' \) extending \( q \) for which there is an edge \( (N_0, \rho_0), (N_1, \rho_1) \in \tau_q \) such that \( \alpha \in \mathcal{C}_{N_0, \rho_0} \), and such that \( \alpha' := \Psi_{N_0, N_1}(\alpha) \) is such that \( \alpha' < \alpha_0 \). The information at the coordinate \( \alpha' \) contained in any extension of \( q' \) will then have to be copied over into the coordinate \( \alpha \), which in this situation means that the real \( r_\alpha \) constructed at coordinate \( \alpha \) is identical to the real at \( \alpha' \), and this obviously prevents \( r_\alpha \) from being Cohen–generic over \( V^{P_{\alpha_0}} \). By the same considerations, the set \( \hat{D}_\alpha \) constructed at a particular coordinate \( \alpha \) of our construction \( (P_\alpha)_{\alpha < \omega_2} \) will typically fail to be a generic club, over \( V^{P_{\alpha_0}} \), for the relevant forcing for measuring the club-sequence \( \hat{C}_\alpha \).

On the other hand, and as we saw in the proof of Lemma 3.16, it will be generic enough to measure \( \hat{C}_\alpha \).
References

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