# "JUST THE MATHS"

# SLIDES NUMBER

# 7.3

# DETERMINANTS 3
# (Further evaluation of 3 x 3 determinants)

## by


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# UNIT 7.3 - DETERMINANTS 3

# FURTHER EVALUATION OF THIRD ORDER DETERMINANTS

## 7.3.1 EXPANSION BY ANY ROW OR COLUMN

**Reminders:**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

This is the **"expansion by the first row"**.

**ILLUSTRATION 1 - Expansion by the second row.**

$$-a_2(b_1c_3 - b_3c_1) + b_2(a_1c_3 - a_3c_1) - c_2(a_1b_3 - a_3b_1)$$

gives exactly the same result as in the original formula.

## ILLUSTRATION 2 - Expansion by the third column

$$c_1(a_2b_3 - a_3b_2) - c_2(a_1b_3 - a_3b_1) + c_3(a_1b_2 - a_2b_1)$$

gives exactly the same result as

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

## Note:

Similar patterns of symbols give the expansions by the remaining rows and columns.

## Summary

A third order determinant may be expanded (that is, evaluated) if we first multiply each of the three elements in any row or (any column) by its minor;

then we combine the results according the following pattern of so-called **"place-signs"**.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}.$$

## COFACTORS

Every **"signed-minor"** is called a **"cofactor"**.

When the place-sign is $+$, the minor and the cofactor are the same.

When the place-sign is $-$, the cofactor is <u>numerically</u> equal to the minor but opposite in sign.

For instance, in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

(i) the minor of $b_1$ is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

but the cofactor of $b_1$ is $-\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

(ii) the minor and cofactor of $b_2$ are both equal to

$$\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}.$$

## 7.3.2 ROW AND COLUMN OPERATIONS ON DETERMINANTS

## INTRODUCTION

The following is especially useful for determinants where some or all of the elements are **variable** quantities.

## STANDARD PROPERTIES OF DETERMINANTS

1. If all of the elements in a row or a column have the value zero, then the value of the determinant is equal to zero.

   **Proof:**

   Expand the determinant by the row or column of zeros.

2. If all but one of the elements in a row or column are equal to zero, then the value of the determinant is the product of the non-zero element in that row or column with its cofactor.

   **Proof:**

   Expand the determinant by the row or column containing the single non-zero element.

   The determinant is effectively equivalent to a determinant of one order lower.

   For example,

$$\begin{vmatrix} 5 & 1 & 0 \\ -2 & 4 & 3 \\ 6 & 8 & 0 \end{vmatrix} = -3\begin{vmatrix} 5 & 1 \\ 6 & 8 \end{vmatrix} = -3(40-6) = -102.$$

3. If a determinant contains two identical rows or two identical columns, then the value of the determinant is zero.

   **Proof:**

   Expand the determinant by a row or column other than the two identical ones.

   All of the cofactors have value zero.

   For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{vmatrix} = -4\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} + 5\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} - 6\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0.$$

4. If two rows, or two columns, are interchanged the value of the determinant is unchanged numerically but it is reversed in sign.

   **Proof:**

   Expand the determinant by a row or column other than the two which have been interchanged

   All of the cofactors will be changed in sign.

For example

$$\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = a_1 \begin{vmatrix} c_2 & b_2 \\ c_3 & b_3 \end{vmatrix} - a_2 \begin{vmatrix} c_1 & b_1 \\ c_3 & b_3 \end{vmatrix} + a_3 \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}.$$

5. If all of the elements in a row or column have a common factor, then this common factor may be removed from the determinant and placed outside.

**Proof:**

Expanding the determinant by the row or column which contains the common factor is equivalent to removing the common factor first, then expanding by the new row or column so created.

For example,

$$\begin{vmatrix} a_1 & kb_1 & c_1 \\ a_2 & kb_2 & c_2 \\ a_3 & kb_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

**Note:**

If all elements in any row or column are multiplied by the same factor, then the value of the determinant is also multiplied by that factor.

6. If the elements of any row in a determinant are altered by adding to them (or subtracting from them) a common multiple of the corresponding elements in another row, then the value of the determinant is unaltered.

   A similar result applies to columns.

   **ILLUSTRATION**

$$\begin{vmatrix} a_1 + kb_1 & b_1 \\ a_2 + kb_2 & b_2 \end{vmatrix} =$$

$$[(a_1 + kb_1)b_2 - (a_2 + kb_2)b_1] = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

## EXAMPLES

Let $R_1$, $R_2$ and $R_3$ denote Row 1, Row 2 and Row 3.

Let $C_1$, $C_2$ and $C_3$ denote Column 1, Column 2 and Column 3.

Let $\longrightarrow$ stand for "becomes".

The following examples use **"row operations"** and **"column operations"**.

1. Evaluate the determinant,

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix}$$

**Solution**

$$\begin{vmatrix} 1 & 15 & 7 \\ 2 & 25 & 9 \\ 3 & 10 & 3 \end{vmatrix} \quad C_1 \quad \longrightarrow \quad C_1 \quad \div \quad 5;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 2 & 5 & 9 \\ 3 & 2 & 3 \end{vmatrix} \quad R_2 \quad \longrightarrow \quad R_2 - 2R_1;$$

$$5\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 3 & 2 & 3 \end{vmatrix} \quad R_3 \quad \longrightarrow \quad R_3 - 3R_1;$$

$$\begin{vmatrix} 1 & 3 & 7 \\ 0 & -1 & -5 \\ 0 & -7 & -18 \end{vmatrix}$$

$$= 5(18 - 35) = 5 \times -17 = -85.$$

2. Solve, for $x$, the equation

$$\begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} = 0.$$

**Solution**

Direct expansion gives a cubic equation in $x$.

Therefore, try to obtain factors of the equation **before** expanding the determinant.

Here, the three expressions in each column add up to the same quantity, namely $x + 2$.

Thus, add Row 2 to Row 1, then add Row 3 to the new Row 1.

This gives $x + 2$ as a factor of the first row.

$$0 = \begin{vmatrix} x & 5 & 3 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2 + R_3$$

$$= \begin{vmatrix} x+2 & x+2 & x+2 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x+2)$$

$$= (x+2)\begin{vmatrix} 1 & 1 & 1 \\ 5 & x+1 & 1 \\ -3 & -4 & x-2 \end{vmatrix}$$

$$C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x+2)\begin{vmatrix} 1 & 0 & 0 \\ 5 & x-4 & -4 \\ -3 & -1 & x+1 \end{vmatrix}$$

$$= (x+2)[(x-4)(x+1)-4] = (x-2)(x^2-3x-8).$$

Hence,

$$x = -2 \text{ or } x = \frac{3 \pm \sqrt{9+32}}{2} = \frac{3 \pm \sqrt{41}}{2}.$$

3. Solve, for $x$, the equation

$$\begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} = 0.$$

**Solution**

The sum of the corresponding pairs of elements in the first two rows is the same, namely $x-4$.

$$0 = \begin{vmatrix} x-6 & -6 & x-5 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 + R_2$$

$$= \begin{vmatrix} x-4 & x-4 & x-4 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix} \quad R_1 \longrightarrow R_1 \div (x-4)$$

$$= (x-4)\begin{vmatrix} 1 & 1 & 1 \\ 2 & x+2 & 1 \\ 7 & 8 & x+7 \end{vmatrix}$$

$$C_2 \longrightarrow C_2 - C_1 \text{ and } C_3 \longrightarrow C_3 - C_1$$

$$= (x-4)\begin{vmatrix} 1 & 0 & 0 \\ 2 & x & -1 \\ 7 & 1 & x \end{vmatrix}$$

$$= (x-4)(x^2+1)$$

$x = 4$ and $x = \pm j$.

4. Solve, for $x$, the equation

$$\begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix}.$$

**Solution**

The 2 in Row 1 may be used to reduce to zero the 4 underneath it in Row 2.

$$0 = \begin{vmatrix} x & 3 & 2 \\ 4 & x+4 & 4 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 - 2R_1$$

$$= \begin{vmatrix} x & 3 & 2 \\ 4-2x & x-2 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad R_2 \longrightarrow R_2 \div (x-2)$$

$$= (x-2)\begin{vmatrix} x & 3 & 2 \\ -2 & 1 & 0 \\ 2 & 1 & x-1 \end{vmatrix} \quad C_1 \longrightarrow C_1 + 2C_2$$

$$= (x-2)\begin{vmatrix} x+6 & 3 & 2 \\ 0 & 1 & 0 \\ 4 & 1 & x-1 \end{vmatrix}$$

$$= (x-2)[(x+6)(x-1)-8] = (x-2)[x^2+5x-14]$$

$$= (x-2)(x+7)(x-2).$$

Thus,

$$x = 2 \text{ (repeated) and } x = -7.$$