"JUST THE MATHS"

SLIDES NUMBER

7.2

DETERMINANTS 2
(Consistency and third order determinants)

by

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UNIT 7.2 - DETERMINANTS 2

CONSISTENCY AND THIRD ORDER DETERMINANTS

7.2.1 CONSISTENCY FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN TWO UNKNOWNS

Consider the set of equations

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, - - - - - - - - - - (1) \\ a_2x + b_2y + c_2 &= 0, - - - - - - - - - - (2) \\ a_3x + b_3y + c_3 &= 0, - - - - - - - - - - - (3) \end{aligned}$$

(Assume that any pair has a unique common solution by Cramer's Rule).

To be consistent, the common solution of any pair must also satisfy the remaining equation.

In particular, the common solution of equations (2) and (3) must also satisfy equation (1).

By Cramer's Rule in equations (2) and (3),

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

This solution will satisfy equation (1) provided that

$$a_1 \frac{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} - b_1 \frac{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}} + c_1 = 0.$$

In other words,

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is the determinant condition for the consistency of three simultaneous linear equations in two unknowns.

7.2.2 THE DEFINITION OF A THIRD ORDER DETERMINANT

In the consistency condition of the previous section, the expression on the left-hand-side is called a **"determinant of the third order"** and is denoted by the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

It has three **"rows"** (horizontally), three **"columns"** (vertically) and nine **"elements"** (the numbers inside the determinant).

The definition of a third order determinant may be stated in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

Notes:

(i) Other forms of the definition are also possible

(ii) Take each element of the first row in turn and multiply by its **"minor"**.

(iii) The Minor is the 2×2 order determinant obtained by covering up the row and column in which the element appears

(iv) the results are then combined according to a $+,\ -,\ +$ pattern.

(v) Rows are counted from the top to the bottom and columns are counted from left to the right.

(vi) Each row is read from the left to the right and each column is read from the top to the bottom.

ILLUSTRATION

The third element of the second column is b_3.

EXAMPLES

1. Evaluate the determinant

$$\Delta = \begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix}.$$

Solution

$$\Delta = -3\begin{vmatrix} 4 & -2 \\ -1 & 3 \end{vmatrix} - 2\begin{vmatrix} 0 & -2 \\ 5 & 3 \end{vmatrix} + 7\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix}.$$

That is,

$$\Delta = -3(12 - 2) - 2(0 + 10) + 7(0 - 20) = -190.$$

2. Show that the simultaneous linear equations

$$\begin{aligned} 3x - y + 2 &= 0, \\ 2x + 5y - 1 &= 0, \\ 5x + 4y + 1 &= 0 \end{aligned}$$

are consistent (assuming that any two of the three have a common solution), and obtain the common solution.

Solution

$$\begin{vmatrix} 3 & -1 & 2 \\ 2 & 5 & -1 \\ 5 & 4 & 1 \end{vmatrix} =$$

$$3\begin{vmatrix} 5 & -1 \\ 4 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 2 & -1 \\ 5 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ 5 & 4 \end{vmatrix}$$

$$= 3(5+4) + (2+5) + 2(8-25) = 27 + 7 - 34 = 0.$$

Thus, the equations are consistent.
Solving the first two:

$$\frac{x}{\begin{vmatrix} -1 & 2 \\ 5 & -1 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & -1 \\ 2 & 5 \end{vmatrix}}.$$

That is,

$$\frac{x}{-9} = \frac{-y}{-7} = \frac{1}{17},$$

which gives

$$x = -\frac{9}{17} \quad \text{and} \quad y = \frac{7}{17}.$$

Notes:

(i) The given set of equations are **"linearly dependent"** (the third equation is the sum of the other two).

(ii) The rows of the determinant of coefficients and constants are linearly dependent
(Row 3 = Row 1 plus Row 2).

(iii) It may shown that the value of a determinant is zero if and only if its rows are linearly dependent.

(iv) An alternative way of proving consistency is to show linear dependence.

7.2.3 THE RULE OF SARRUS

So far,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} =$$

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

But the same terms can be obtained from following diagram:

$$\left|\begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix}\right| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix}.$$

Take the sum of the possible products of the trios of numbers in the direction $\searrow$

and subtract the sum of the possible products of the trios of numbers in the $\nearrow$ direction:

$$(a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_2c_3a_1 + c_3a_2b_1).$$

These are exactly the same terms as those obtained by the original formula.

The **"Rule of Sarrus"** is ideal for electronic calculators.

EXAMPLE

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} =$$

$$\begin{vmatrix} -3 & 2 & 7 \\ 0 & 4 & -2 \\ 5 & -1 & 3 \end{vmatrix} \begin{matrix} -3 & 2 \\ 0 & 4 \\ 5 & -1 \end{matrix}$$

$$= ([-3].4.3 + 2.[-2].5 + 7.0.[-1])$$

$$-(5.4.7 + [-1].[-2].[-3] + 3.0.2)$$

$$= (-36 - 20 + 0) - (140 - 6 + 0)$$

$$= -56 - 134 = -190.$$

7.2.4 CRAMER'S RULE FOR THREE SIMULTANEOUS LINEAR EQUATIONS IN THREE UNKNOWNS

$$\begin{aligned} a_1x + b_1y + c_1z + d_1 &= 0, \\ a_2x + b_2y + c_2z + d_2 &= 0, \\ a_3x + b_3y + c_3z + d_3 &= 0, \end{aligned}$$

have a common solution, given by

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

which is called the "Key" to the solution and requires that $\Delta_0 \neq 0$.

Again the rule is known as "Cramer's Rule".

EXAMPLE

Using the Rule of Sarrus, obtain the common solution of the simultaneous linear equations

$$\begin{aligned} x + 4y - z + 2 &= 0, \\ -x - y + 2z - 9 &= 0, \\ 2x + y - 3z + 15 &= 0. \end{aligned}$$

Solution

The "Key" is

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_0},$$

where

(i)

$$\Delta_0 = \left| \begin{array}{ccc} 1 & 4 & -1 \\ -1 & -1 & 2 \\ 2 & 1 & -3 \end{array} \right| \begin{array}{cc} 1 & 4 \\ -1 & -1 \\ 2 & 1 \end{array}.$$

Hence,

$$\Delta_0 = (3 + 16 + 1) - (2 + 2 + 12) = 20 - 16 = 4$$

which is non-zero, and so we may continue:

(ii)

$$\Delta_1 = \left|\begin{array}{ccc|cc} 4 & -1 & 2 & 4 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 1 & -3 & 15 & 1 & -3 \end{array}\right. .$$

Hence,

$$\Delta_1 = (120 + 9 + 6) - (4 + 108 + 15) = 135 - 127 = 8.$$

(iii)

$$\Delta_2 = \left|\begin{array}{ccc|cc} 1 & -1 & 2 & 1 & -1 \\ -1 & 2 & -9 & -1 & 2 \\ 2 & -3 & 15 & 2 & -3 \end{array}\right. .$$

Hence,

$$\Delta_2 = (30 + 18 + 6) - (8 + 27 + 15) = 54 - 50 = 4.$$

(iv)

$$\Delta_3 = \left|\begin{array}{ccc|cc} 1 & 4 & 2 & 1 & 4 \\ -1 & -1 & -9 & -1 & -1 \\ 2 & 1 & 15 & 2 & 1 \end{array}\right. .$$

Hence,

$$\Delta_3 = (-15 - 72 - 2) - (-4 - 9 - 60) = -89 + 73 = -16.$$

(v) The solutions are therefore

$$x = -\frac{\Delta_1}{\Delta_0} = -\frac{8}{4} = -2;$$

$$y = \frac{\Delta_2}{\Delta_0} = \frac{4}{4} = 1;$$

$$z = -\frac{\Delta_3}{\Delta_0} = -\frac{-16}{4} = 4.$$

Special Cases

If it should happen that $\Delta_0 = 0$, the rows of Δ_0 must be linearly dependent.

That is, the three groups of x, y and z terms must be linearly dependent.

Different situations arise according to the constant terms:

EXAMPLES

1. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 4 &= 0, \end{aligned}$$

the third equation is the difference between the first two and hence it is redundant.

There will be an infinite number of solutions. For example, we may choose z at random, solving for x and y;

$$x = \frac{11 - 5z}{5} \quad \text{and} \quad y = \frac{5z - 3}{5}.$$

2. For the simultaneous linear equations

$$\begin{aligned} 2x - y + 3z - 5 &= 0, \\ x + 2y - z - 1 &= 0, \\ x - 3y + 4z - 7 &= 0, \end{aligned}$$

the third equation is inconsistent with the difference between the first two equations. That is,

$x - 3y + 4z - 7 = 0$ inconsistent with $x - 3y + 4z - 4 = 0$.

In this case, there are no common solutions.

3. For the simultaneous linear equations

$$\begin{aligned} x - 2y + 3z - 1 &= 0, \\ 2x - 4y + 6z - 2 &= 0, \\ 3x - 6y + 9z - 3 &= 0 \end{aligned}$$

we have only one independent equation.

There will be an infinite number of solutions.

Choose two of the variables at random, then determine the remaining variable.

Summary of the special cases

If $\Delta_0 = 0$, further investigation of the simultaneous linear equations is necessary.