"JUST THE MATHS"

SLIDES NUMBER

4.2

HYPERBOLIC FUNCTIONS 2
(Inverse hyperbolic functions)

by

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UNIT 4.2 - HYPERBOLIC FUNCTIONS 2 INVERSE HYPERBOLIC FUNCTIONS

4.2.1 - INTRODUCTION

The three basic inverse hyperbolic functions are $\text{Cosh}^{-1}x$, $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$.

It may be shown that

(a)

$$\text{Cosh}^{-1}x = \pm \ln(x + \sqrt{x^2 - 1}).$$

(b)

$$\text{Sinh}^{-1}x = \ln(x + \sqrt{x^2 + 1}).$$

(c)

$$\text{Tanh}^{-1}x = \frac{1}{2} \ln \frac{1 + x}{1 - x}.$$

Notes:

(i) The positive value of $\text{Cosh}^{-1}x$ is called the **"principal value"** and is denoted by $\cosh^{-1}x$ (using a lower-case c).

(ii) $\text{Sinh}^{-1}x$ and $\text{Tanh}^{-1}x$ have only **one** value but, for uniformity, we denote them by $\sinh^{-1}x$ and $\tanh^{-1}x$

4.2.2 THE PROOFS OF THE STANDARD FORMULAE

(a) Inverse Hyperbolic Cosine

If we let $y = \text{Cosh}^{-1}x$, then

$$x = \cosh y = \frac{e^y + e^{-y}}{2}.$$

Hence,

$$2x = e^y + e^{-y}.$$

On rearrangement,

$$(e^y)^2 - 2xe^y + 1 = 0.$$

Hence,

$$e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}.$$

Taking natural logarithms

$$y = \ln(x \pm \sqrt{x^2 - 1}) = \pm \ln(x + \sqrt{x^2 - 1}),$$

since $x + \sqrt{x^2 - 1}$ and $x - \sqrt{x^2 - 1}$ are reciprocals of each other, their product being the value 1.

(b) Inverse Hyperbolic Sine

If we let $y = \text{Sinh}^{-1}x$, then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}.$$

Hence,

$$2x = e^y - e^{-y}$$

or

$$(e^y)^2 - 2xe^y - 1 = 0.$$

Hence,

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

However, $x - \sqrt{x^2 + 1} < 0$ and cannot, therefore, be equated to a power of e.

Taking natural logarithms,

$$y = \ln(x + \sqrt{x^2 + 1}).$$

(c) Inverse Hyperbolic Tangent

If we let $y = \text{Tanh}^{-1}x$, then

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} = \frac{e^{2y} - 1}{e^{2y} + 1}.$$

Hence,

$$x\left(e^{2y} + 1\right) = e^{2y} - 1,$$

giving

$$e^{2y} = \frac{1 + x}{1 - x}.$$

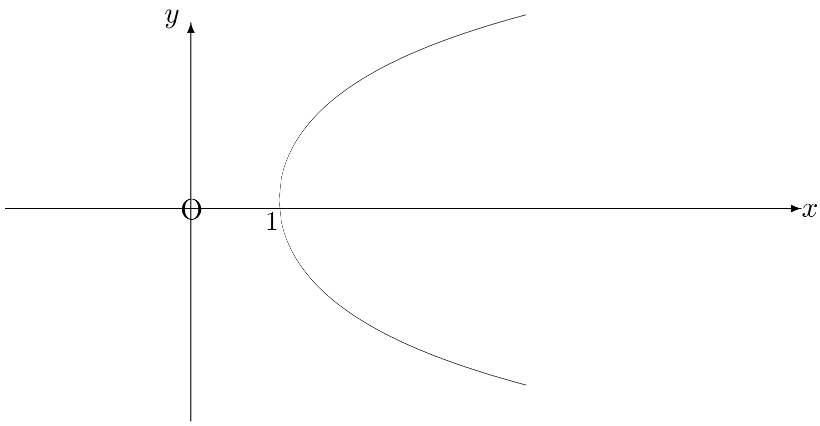
Taking natural logarithms,

$$y = \frac{1}{2}\ln\frac{1 + x}{1 - x}.$$

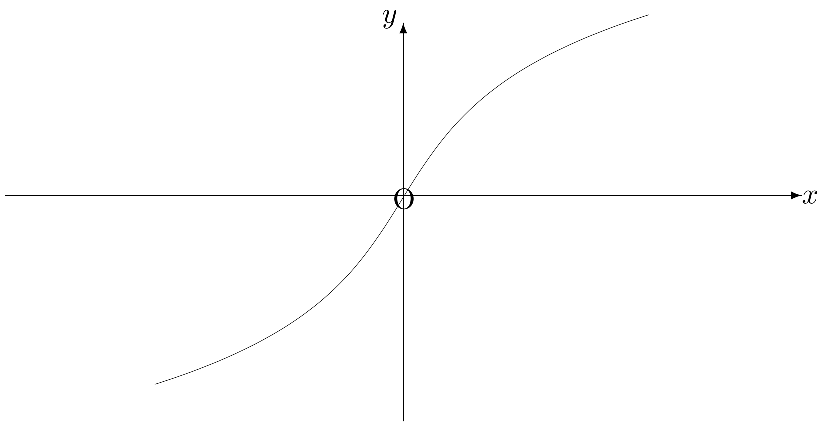
Note:
The graphs of inverse hyperbolic functions are discussed fully in Unit 10.7, but we include them here for the sake of completeness:

The graphs are as follows:

(a) $y = \text{Cosh}^{-1}x$



(b) $y = \mathrm{Sinh}^{-1} x$



(c) $y = \mathrm{Tanh}^{-1} x$