“JUST THE MATHS”

SLIDES NUMBER

4.1

HYPERBOLIC FUNCTIONS 1
(Definitions, graphs and identities)

by

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UNIT 4.1 - HYPERBOLIC FUNCTIONS 1 DEFINTIONS, GRAPHS AND IDENTITIES

4.1.1 INTRODUCTION

We introduce a new group of mathematical functions, based on the functions

$$e^x \text{ and } e^{-x}.$$

Their properties resemble, very closely, those of the standard trigonometric functions.

Just as trigonometric functions can be related to the geometry of a circle, the new functions can be related to the geometry of a **hyperbola**.

4.1.2 DEFINITIONS

(a) Hyperbolic Cosine

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}.$$

The name of the function is pronounced **"cosh"**.

(b) Hyperbolic Sine

$$\sinh x \equiv \frac{e^x - e^{-x}}{2}.$$

The name of the function is pronounced **"shine"**.

(c) Hyperbolic Tangent

$$\tanh x \equiv \frac{\sinh x}{\cosh x}.$$

The name of the function is pronounced **than"**.

In terms of exponentials, it is easily shown that

$$\tanh x \equiv \frac{e^x - e^{-x}}{e^x + e^{-x}} \equiv \frac{e^{2x} - 1}{e^{2x} + 1}.$$

(d) Other Hyperbolic Functions

(i) Hyperbolic secant , pronounced **"shek"**.

$$\text{sech} x \equiv \frac{1}{\cosh x}.$$

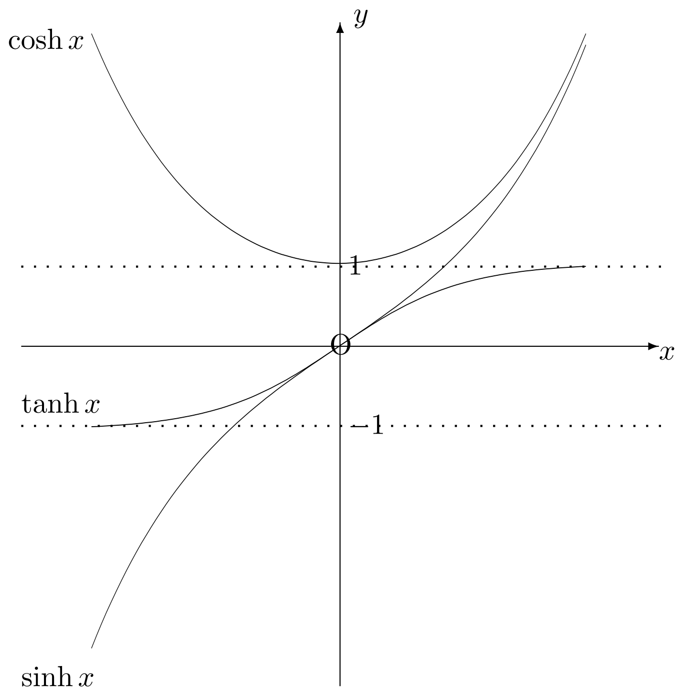
(ii) Hyperbolic cosecant , pronounced **'coshek"**.

$$\text{cosech} x \equiv \frac{1}{\sinh x}.$$

(iii) Hyperbolic cotangent , pronounced **"coth"**.

$$\text{coth} x \equiv \frac{1}{\tanh x} \equiv \frac{\cosh x}{\sinh x}.$$

4.1.3 GRAPHS OF HYPERBOLIC FUNCTIONS



The graph of $\cosh x$ exists only for y greater than or equal to 1.

The graph of $\tanh x$ exists only for y lying between -1 and $+1$.

The graph of $\sinh x$ covers the whole range of x and y values from $-\infty$ to $+\infty$.

4.1.4 HYPERBOLIC IDENTITIES

For every identity obeyed by trigonometric functions, there is a corresponding identity obeyed by hyperbolic functions.

ILLUSTRATIONS

1.

$$e^x \equiv \cosh x + \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} \equiv e^x.$$

2.

$$e^{-x} \equiv \cosh x - \sinh x.$$

Proof

$$\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} \equiv e^{-x}.$$

3.

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

Proof

Multiply together the results of the previous two illustrations;

$$e^x . e^{-x} = 1;$$

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \cosh^2 x - \sinh^2 x.$$

Notes:

(i) Dividing throughout by $\cosh^2 x$ gives the identity,

$$1 - \tanh^2 x \equiv \text{sech}^2 x.$$

(ii) Dividing throughout by $\sinh^2 x$ gives the identity,

$$\coth^2 x - 1 \equiv \text{cosech}^2 x.$$

4.

$$\sinh(x + y) \equiv \sinh x \cosh y + \cosh x \sinh y.$$

Proof:

The right hand side is

$$\frac{e^x - e^{-x}}{2} . \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} . \frac{e^y - e^{-y}}{2}.$$

That is,

$$\frac{e^{(x+y)} + e^{(x-y)} - e^{(-x+y)} - e^{(-x-y)}}{4}$$

$$+\frac{e^{(x+y)} - e^{(x-y)} + e^{(-x+y)} - e^{(-x-y)}}{4}.$$

This simplifies to

$$\frac{2e^{(x+y)} - 2e^{(-x-y)}}{4}.$$

That is,

$$\frac{e^{(x+y)} - e^{-(x+y)}}{2} \equiv \sinh(x+y).$$

5.

$$\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y.$$

Proof

The proof is similar to Illustration 4.

6.

$$\tanh(x+y) \equiv \frac{\tanh x + \tanh y}{1 - \tanh x \tanh y}.$$

Proof

The proof is similar to Illustration 4.

4.1.5 OSBORN'S RULE

Starting with any trigonometric identity, change cos to cosh and sin to sinh.

Then, if the trigonometric identity contains (or implies) two sine functions multiplied together, change the sign in front of the relevant term from + to − or vice versa.

ILLUSTRATIONS

1.

$$\cos^2 x + \sin^2 x \equiv 1$$

leads to

$$\cosh^2 x - \sinh^2 x \equiv 1.$$

2.

$$\sin(x - y) \equiv \sin x \cos y - \cos x \sin y$$

leads to

$$\sinh(x - y) \equiv \sinh x \cosh y - \cosh x \sinh y.$$

3.

$$\sec^2 x \equiv 1 + \tan^2 x$$

leads to

$$\text{sech}^2 x \equiv 1 - \tanh^2 x.$$