“JUST THE MATHS”

SLIDES NUMBER

2.3

SERIES 3
(Elementary convergence and divergence)

by

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UNIT 2.3 - ELEMENTARY CONVERGENCE AND DIVERGENCE

Introduction

An infinite series may be specified by either

$$u_1 + u_2 + u_3 + \ldots = \sum_{r=1}^{\infty} u_r$$

or

$$u_0 + u_1 + u_2 + \ldots = \sum_{r=0}^{\infty} u_r.$$

In the first of these, u_r is the r-th term while, in the second, u_r is the $(r+1)$-th term.

ILLUSTRATIONS

1.
$$1 + \frac{1}{2} + \frac{1}{3} + \ldots = \sum_{r=1}^{\infty} \frac{1}{r} = \sum_{r=0}^{\infty} \frac{1}{r+1}.$$

2.
$$2 + 4 + 6 + 8 + \ldots = \sum_{r=1}^{\infty} 2r = \sum_{r=0}^{\infty} 2(r+1).$$

3.
$$1 + 3 + 5 + 7 + \ldots = \sum_{r=1}^{\infty} (2r-1) = \sum_{r=0}^{\infty} (2r+1).$$

2.3.1 THE DEFINITIONS OF CONVERGENCE AND DIVERGENCE

An infinite series may have a **"sum to infinity"** even though it is not possible to reach the end of the series.

For example, in

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots \; = \sum_{r=1}^{\infty} \frac{1}{2^r},$$

$$S_n = \frac{\frac{1}{2}(1 - \frac{1}{2}^n)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}.$$

As n becomes larger and larger, S_n approaches 1.

We say that the **"limiting value"** of S_n as n **"tends to infinity"** is 1; and we write

$$\lim_{n \to \infty} S_n = 1.$$

Since this limiting value is a **finite** number, we say that the series **"converges"** to 1.

DEFINITION (A)

For the infinite series

$$\sum_{r=1}^{\infty} u_r,$$

the expression

$$u_1 + u_2 + u_3 + \ldots + u_n$$

is called its **"n-th partial sum"**.

DEFINITION (B)

If the n-th Partial Sum of an infinite series tends to a finite limit as n tends to infinity, the series is said to **"converge"**. In **all** other cases, the series is said to **"diverge"**.

ILLUSTRATIONS

1.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = \sum_{r=1}^{\infty} \frac{1}{2^r} \text{ converges.}$$

2.

$$1 + 2 + 3 + 4 + \ldots. = \sum_{r=1}^{\infty} r \text{ diverges.}$$

3.

$$1 - 1 + 1 - 1 + \ldots. = \sum_{r=1}^{\infty} (-1)^{n-1} \text{ diverges.}$$

Notes:

(i) Illustration 3 shows that a series which diverges does not necessarily diverge to infinity.

(ii) Whether a series converges or diverges depends less on the starting terms than it does on the later terms.

For example

$$7 - 15 + 2 + 39 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ . \ .$$

converges to $7 - 15 + 2 + 39 + 1 = 33 + 1 = 34$.

(iii) It is sometimes possible to test an infinite series for convergence or divergence without having to determine its sum to infinity.

2.3.2 TESTS FOR CONVERGENCE AND DIVERGENCE

First, we shall consider series of **positive** terms only.

TEST 1 - The r-th Term Test

An infinite series,

$$\sum_{r=1}^{\infty} u_r,$$

cannot converge unless

$$\lim_{r \to \infty} u_r = 0.$$

Outline Proof:

The series will converge only if the r-th partial sums, S_r, tend to a finite limit, L (say), as r tends to infinity.

Since $u_r = S_r - S_{r-1}$, then u_r must tend to $L - L = 0$ as r tends to infinity.

ILLUSTRATIONS

1. The convergent series

$$\sum_{r=1}^{\infty} \frac{1}{2^r}$$

is such that

$$\lim_{r\to\infty} \frac{1}{2^r} = 0.$$

2. The divergent series

$$\sum_{r=1}^{\infty} r$$

is such that

$$\lim_{r\to\infty} r \neq 0.$$

3. The series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is such that

$$\lim_{r\to\infty} \frac{1}{r} = 0,$$

but this series is **divergent** (see later).

N.B. The converse of the r-th Term Test is not true.

TEST 2 - The Comparison Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\sum_{r=1}^{\infty} v_r$$

is a second series which is known to **converge**.

Then the first series converges provided that $u_r \leq v_r$.

Similarly, suppose

$$\sum_{r=1}^{\infty} w_r$$

is a series which is known to **diverge**

Then the first series diverges provided that $u_r \geq w_r$.

Note:
It may be necessary to ignore the first few values of r.

Outline Proof of Comparison Test:

Think of u_r and v_r as the heights of two sets of rectangles, all with a common base-length of one unit.

(i) If the series

$$\sum_{r=1}^{\infty} v_r$$

is **convergent** it represents a **finite** total area of an infinite number of rectangles.

The series

$$\sum_{r=1}^{\infty} u_r$$

represents a **smaller** area and, hence, is also finite.

(ii) A similar argument holds when

$$\sum_{r=1}^{\infty} w_r$$

is a **divergent** series and $u_r \geq w_r$.

A divergent series of **positive** terms can diverge only to $+\infty$ so that the set of rectangles determined by u_r generates an area that is greater than an area which is already infinite.

EXAMPLES

1. Show that the series

$$\sum_{r=1}^{\infty} \frac{1}{r} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ . \ .$$

is divergent.

Solution

The given series may be written as

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ . \ . \ .$$

a series whose terms are all $\geq \frac{1}{2}$.

But the series

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ . \ .$$

is a divergent series and, hence, the series

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent.

2. Given that

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is a convergent series, show that

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

is also a convergent series.

Solution

Firstly, for $r = 1, 2, 3, 4, \ldots,$

$$\frac{1}{r(r+1)} < \frac{1}{r.r} = \frac{1}{r^2}.$$

Hence the terms of the series

$$\sum_{r=1}^{\infty} \frac{1}{r(r+1)}$$

are smaller in value than those of a known convergent series. It therefore converges also.

Note:

It may be shown that the series

$$\sum_{r=1}^{\infty} \frac{1}{r^p}$$

is convergent whenever $p > 1$ and divergent whenever $p \leq 1$.

TEST 3 - D'Alembert's Ratio Test

Given the series

$$\sum_{r=1}^{\infty} u_r,$$

suppose that

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = L;$$

Then the series converges if $L < 1$ and diverges if $L > 1$.

There is no conclusion if $L = 1$.

Outline Proof:

(i) If $L > 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be greater than 1.

Thus, $u_{r+1} > u_r$ for a large enough value of r.

Hence, the terms cannot ultimately be decreasing; so Test 1 shows that the series diverges.

(ii) If $L < 1$, **all** the values of $\frac{u_{r+1}}{u_r}$ will **ultimately** be less than 1.

Thus, $u_{r+1} < u_r$ for a large enough value or r.

Let this occur first when $r = s$.

From this value onwards, the terms steadily decrease in value.

We can certainly find a positive number, h, between L and 1 such that

$$\frac{u_{s+1}}{u_s} < h, \frac{u_{s+2}}{u_{s+1}} < h, \frac{u_{s+3}}{u_{s+2}} < h, \ \ldots$$

That is,

$$u_{s+1} < hu_s, \ u_{s+2} < hu_{s+1}, \ u_{s+3} < hu_{s+2}, \ \ldots,$$

which gives

$$u_{s+1} < hu_s, \ u_{s+2} < h^2u_s, \ u_{s+3} < h^3u_s, \ \ldots$$

But, since $L < h < 1$,

$$hu_s + h^2u_s + h^3u_s + \ \ldots$$

is a convergent geometric series.

Therefore, by the Comparison Test,

$$u_{s+1} + u_{s+2} + u_{s+3} + \ \ldots = \sum_{r=1}^{\infty} \ u_{s+r} \ \text{ converges.}$$

(iii) If $L = 1$, there will be no conclusion since we have already encountered examples of both a convergent series **and** a divergent series which give $L = 1$.

In particular,

$$\sum_{r=1}^{\infty} \frac{1}{r}$$

is divergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r}{r+1} = \lim_{r \to \infty} \frac{1}{1 + \frac{1}{r}} = 1.$$

Also,

$$\sum_{r=1}^{\infty} \frac{1}{r^2}$$

is convergent and gives

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} \frac{r^2}{(r+1)^2} = \lim_{r \to \infty} \left(\frac{r}{r+1}\right)^2$$

$$= \lim_{r \to \infty} \left(\frac{1}{1 + \frac{1}{r}}\right)^2 = 1$$

Note:

To calculate the limit as r tends to infinity of any ratio of two polynomials in r, divide the numerator and the denominator by the highest power of r.

For example,

$$\lim_{r\to\infty} \frac{3r^3+1}{2r^3+1} = \lim_{r\to\infty} \frac{3+\frac{1}{r^3}}{2+\frac{1}{r^3}} = \frac{3}{2}.$$

ILLUSTRATIONS

1. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} \frac{r}{2^r},$$

$$\frac{u_{r+1}}{u_r} = \frac{r+1}{2^{r+1}}.\frac{2^r}{r} = \frac{r+1}{2r}.$$

Thus,

$$\lim_{r\to\infty} \frac{u_{r+1}}{u_r} = \lim_{r\to\infty} \frac{r+1}{2r} = \lim_{r\to\infty} \frac{1+\frac{1}{r}}{2} = \frac{1}{2}.$$

The limiting value is less than 1 so that the series converges.

2. For the series

$$\sum_{r=1}^{\infty} u_r = \sum_{r=1}^{\infty} 2^r,$$

$$\frac{u_{r+1}}{u_r} = \frac{2^{r+1}}{2^r} = 2.$$

Thus,

$$\lim_{r \to \infty} \frac{u_{r+1}}{u_r} = \lim_{r \to \infty} 2 = 2.$$

The limiting value is greater than 1 so that the series diverges.