"JUST THE MATHS"

SLIDES NUMBER

2.2

SERIES 2
(Binomial series)

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UNIT 2.2 - SERIES 2 - BINOMIAL SERIES

INTRODUCTION

In this section, we expand (multiplying out) an expression of the form

$$(A + B)^n.$$

A and B can be either mathematical expressions or numerical values.

n is a given number which need not be a positive integer.

2.2.1 PASCAL'S TRIANGLE

ILLUSTRATIONS

1. $(A + B)^1 \equiv$

$$A + B.$$

2. $(A + B)^2 \equiv$

$$A^2 + 2AB + B^2.$$

3. $(A + B)^3 \equiv$

$$A^3 + 3A^2B + 3AB^2 + B^3.$$

4. $(A + B)^4 \equiv$

$$A^4 + 4A^3B + 6A^2B^2 + 4AB^3 + B^4.$$

OBSERVATIONS

(i) The expansions begin with the maximum possible power of A and end with the maximum possible power of B.

(ii) The powers of A **decrease** in steps of 1 while the powers of B **increase** in steps of 1.

(iii) The coefficients follow the diagramatic pattern called

PASCAL'S TRIANGLE:

$$1 \quad 1$$

$$1 \quad 2 \quad 1$$

$$1 \quad 3 \quad 3 \quad 1$$

$$1 \quad 4 \quad 6 \quad 4 \quad 1$$

Each line begins and ends with the number 1.

Each of the other numbers is the sum of the two numbers above it in the previous line.

The next line would be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1$$

5. $(A + B)^5 \equiv$

$$A^5 + 5A^4B + 10A^3B^2 + 10A^2B^3 + 5AB^4 + B^5.$$

(iv) For

$$(A - B)^n$$

the terms are alternately positive and negative.

6. $(A - B)^6 \equiv$

$$A^6 - 6A^5B + 15A^4B^2 - 20A^3B^3 + 15A^2B^4 - 6AB^5 + B^6.$$

2.2.2 BINOMIAL FORMULAE

A more general method which can be applied to any value of n is the binomial formula.

DEFINITION

If n is a positive integer, the product

$$1.2.3.4.5..........n$$

is denoted by the symbol $n!$ and is called **"n factorial"**.

Note:

This definition could not be applied to the case when $n = 0$.

$0!$ is defined separately by the statement

$$0! = 1.$$

There is no meaning to $n!$ when n is a negative integer.

(a) Binomial formula for $(A+B)^n$ when n is a positive integer.

It can be shown that

$$(A+B)^n \equiv$$

$$A^n + nA^{n-1}B + \frac{n(n-1)}{2!}A^{n-2}B^2 +$$

$$\frac{n(n-1)(n-2)}{3!}A^{n-3}B^3 + ... + B^n.$$

Notes:

(i) This is the same result as given by Pascal's Triangle.

(ii) The last term is

$$\frac{n(n-1)(n-2)(n-3).......3.2.1}{n!}A^{n-n}B^n = A^0B^n = B^n.$$

(iii) The coefficient of $A^{n-r}B^r$ in the expansion is

$$\frac{n(n-1)(n-2)(n-3).......(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}$$

and this is sometimes denoted by the symbol $\binom{n}{r}$.

(iv) A commonly used version is

$$(1+x)^n \equiv 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+...+x^n.$$

EXAMPLES

1. Expand fully the expression $(1+2x)^3$.

   Solution

   $$(A+B)^3 \equiv A^3+3A^2B+3AB^2+B^3.$$

   Replace A by 1 and B by $2x$.

   $$(1+2x)^3 \equiv 1+3(2x)+3(2x)^2+(2x)^3$$

   $$\equiv 1+6x+12x^2+8x^3.$$

2. Expand fully the expression $(2-x)^5$.

   Solution

   $$(A+B)^5 \equiv A^5+5A^4B+10A^3B^2+10A^2B^3+5AB^4+B^5.$$

   Replace A by 2 and B by $-x$.

   $$(2-x)^5 \equiv 2^5+5(2)^4(-x)+10(2)^3(-x)^2+$$

   $$10(2)^2(-x)^3+5(2)(-x)^4+(-x)^5.$$

   That is,

   $$(2-x)^5 \equiv 32-80x+80x^2-40x^3+10x^4-x^5.$$

(b) Binomial formula for $(A+B)^n$ when n is negative or a fraction.

This time, the series will be an **infinite** series.

RESULT

If n is negative or a fraction and x lies strictly between $x=-1$ and $x=1$, it can be shown that

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+......$$

EXAMPLES

1. Expand $(1+x)^{\frac{1}{2}}$ as far as the term in x^3.

   Solution

   $$(1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^3+......$$

   $$= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-......$$

   provided $-1 < x < 1$.

2. Expand $(2-x)^{-3}$ as far as the term in x^3 stating the values of x for which the series is valid.

Solution

First convert the expression $(2-x)^{-3}$ to one in which the leading term in the bracket is 1.

$$(2-x)^{-3} \equiv \left[2\left(1-\frac{x}{2}\right)\right]^{-3}$$

$$\equiv \frac{1}{8}\Big(1+\Big[-\frac{x}{2}\Big]\Big)^{-3}.$$

The required binomial expansion is

$$\frac{1}{8}[1+(-3)\left(-\frac{x}{2}\right)+\frac{(-3)(-3-1)}{2!}\Big(-\frac{x}{2}\Big)^2+$$

$$\frac{(-3)(-3-1)(-3-2)}{3!}\Big(-\frac{x}{2}\Big)^3+......].$$

That is,

$$\frac{1}{8}\left[1+\frac{3x}{2}+\frac{3x^2}{2}+\frac{5x^3}{4}+......\right].$$

The expansion is valid provided $-x/2$ lies strictly between -1 and 1.

Hence, $-2<x<2$.

(c) Approximate Values

The Binomial Series may be used to calculate simple approximations, as illustrated by the following example:

EXAMPLE

Evaluate $\sqrt{1.02}$ correct to five places of decimals.

Solution

Using $1.02 = 1 + 0.02$, we may say that

$$\sqrt{1.02} = (1 + 0.02)^{\frac{1}{2}}.$$

That is,

$$\sqrt{1.02} = 1 + \frac{1}{2}(0.02) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{1.2}(0.02)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{1.2.3}(0.02)^3 + \ldots$$

$$= 1 + 0.01 - \frac{1}{8}(0.0004) + \frac{1}{16}(0.000008) - \ldots$$

$$= 1 + 0.01 - 0.00005 + 0.0000005 - \ \ldots$$

$$\simeq 1.010001 - 0.000050 = 1.009951$$

Hence, $\sqrt{1.02} \simeq 1.00995$