"JUST THE MATHS"

SLIDES NUMBER

2.1

SERIES 1 (Elementary progressions and series)

by

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UNIT 2.1 - SERIES 1
ELEMENTARY PROGRESSIONS AND SERIES

2.1.1 ARITHMETIC PROGRESSIONS

The **"sequence"** of numbers,

$$a, a + d, a + 2d, a + 3d, ...$$

is said to form an **"arithmetic progression"**.

The symbol a represents the **"first term"**.

The symbol d represents the **"common difference"**

The **"n-th term"** is given by the expression

$$a + (n - 1)d.$$

EXAMPLES

1. Determine the n-th term of the arithmetic progression

$$15, 12, 9, 6, ...$$

Solution

The n-th term is

$$15 + (n - 1)(-3) = 18 - 3n.$$

2. Determine the n-th term of the arithmetic progression

$$8, 8.125, 8.25, 8.375, 8.5, ...$$

Solution

The n-th term is

$$8 + (n - 1)(0.125) = 7.875 + 0.125n.$$

3. The 13th term of an arithmetic progression is 10 and the 25th term is 20; calculate

(a) the common difference;

(b) the first term;

(c) the 17th term.

Solution

$$a + 12d = 10$$

and

$$a + 24d = 20.$$

Hence,

(a) $12d = 10$, so $d = \frac{10}{12} = \frac{5}{6} \simeq 0.83$

(b) $a + 12 \times \frac{5}{6} = 10$, so $a + 10 = 10$ and $a = 0$.

(c) 17th term $= 0 + 16 \times \frac{5}{6} = \frac{80}{6} = \frac{40}{3} \simeq 13.3$

2.1.2 ARITHMETIC SERIES

If the terms of an arithmetic progression are added together, we obtain an **"arithmetic series"**

The total sum of the first n terms is denoted by S_n.

$S_n = a+[a+d]+[a+2d]+...+[a+(n-2)d]+[a+(n-1)d].$

TRICK

Write down the formula **forwards and backwards**.

$S_n = a+[a+d]+[a+2d]+...+[a+(n-2)d]+[a+(n-1)d].$

$S_n = [a+(n-1)d]+[a+(n-2)d]+...+[a+2d]+[a+d]+a.$

Adding gives $2S_n =$

$$[2a + (n-1)d] + ... + [2a + (n-1)d] + [2a + (n-1)d].$$

R.H.S gives n repetitions of the same expression.

Hence,

$$2S_n = n[2a + (n-1)d]$$

or

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

Alternatively,

$$S_n = \frac{n}{2}[\text{FIRST} \quad + \quad \text{LAST}].$$

This is n times the average of the first and last terms.

EXAMPLES

1. Determine the sum of the natural numbers from 1 to 100.

   Solution

   The sum is given by

   $$\frac{100}{2} \times [1 + 100] = 5050.$$

2. How many terms of the arithmetic series

   $$10 + 12 + 14 + ...$$

   must be taken so that the sum of the series is 252 ?

   Solution

   The first term is clearly 10 and the common difference is 2.

   If n is the required number of terms,

$$252 = \frac{n}{2}[20 + (n-1)\times 2];$$

$$252 = \frac{n}{2}[2n + 18] = n(n+9);$$

$$n^2 + 9n - 252 = 0$$

or

$$(n-12)(n+21) = 0;$$

$n = 12$ ignoring $n = -21$.

3. A contractor agrees to sink a well 250 metres deep at a cost of £2.70 for the first metre, £2.85 for the second metre and an extra 15p for each additional metre. Find the cost of the last metre and the total cost.

   Solution

   We need an arithmetic series of 250 terms whose first term is 2.70 and whose common difference is 0.15

   The cost of the last metre is the 250-th term.

   Cost of last metre = £$[2.70 + 249 \times 0.15]$ = £40.05

   The total cost = £$\frac{250}{2} \times [2.70 + 40.05]$ = £5343.75

2.1.3 GEOMETRIC PROGRESSIONS

The sequence of numbers

$$a, ar, ar^2, ar^3, ...$$

is said to form a **"geometric progression"**.

The symbol a represents the **"first term"**.

The symbol r represents the **"common ratio"**.

The **"n-th term"** is given by the expression

$$ar^{n-1}.$$

EXAMPLES

1. Determine the n-th term of the geometric progression

   $$3, -12, 48, -192, ...$$

   Solution

   The n-th term is

   $$3(-4)^{n-1}.$$

   This will be positive when n is an odd number and negative when n is an even number.

2. Determine the seventh term of the geometric progression

   $$3, 6, 12, 24, ...$$

   Solution

The seventh term is

$$3(2^6) = 192.$$

3. The third term of a geometric progression is 4.5 and the ninth is 16.2. Determine the common ratio.

Solution

$$ar^2 = 4.5$$

and

$$ar^8 = 16.2$$

$$\frac{ar^8}{ar^2} = \frac{16.2}{4.5}$$

$$r^6 = 3.6$$

$$r = \sqrt[6]{3.6} \simeq 1.238$$

4. The expenses of a company are £200,000 a year. It is decided that each year they shall be reduced by 5% of those for the preceding year.
What will be the expenses during the fourth year, the first reduction taking place at the end of the first year.

Solution

We use a geometric progression with first term
200,000 and common ratio 0.95

The expenses during the fourth year will be the fourth term of the progression.
Expenses in fourth year = £200,000 $\times (0.95)^3$ = £171475.

2.1.4 GEOMETRIC SERIES

If the terms of a geometric progression are added together, we obtain what is called a **"geometric series"**.

The total sum of a geometric series with n terms is denoted by the S_n.

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}.$$

TRICK

Write down both S_n and rS_n.

$$S_n = a + ar + ar^2 + ar^3 + ... + ar^{n-1}$$
$$rS_n = ar + ar^2 + ar^3 + ar^4 + ... + ar^{n-1} + ar^n;$$

$$S_n - rS_n = a - ar^n;$$

$$S_n = \frac{a(1 - r^n)}{1 - r}.$$

Alternatively (eg. when $r > 1$)

$$S_n = \frac{a(r^n - 1)}{r - 1}.$$

EXAMPLES

1. Determine the sum of the geometric series

$$4 + 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}.$$

Solution

The sum is given by

$$S_6 = \frac{4(1 - (\frac{1}{2})^6)}{1 - \frac{1}{2}} = \frac{4(1 - 0.0156)}{0.5} \simeq 7.875$$

2. A sum of money £C is invested for n years at an interest of $100r\%$, compounded annually. What will be the total interest earned by the end of the n-th year ?

Solution

At the end of year 1, the interest earned will be Cr.

At the end of year 2, the interest earned will be $(C + Cr)r = Cr(1 + r)$.

At the end of year 3, the interest earned will be $C(1 + r)r + C(1 + r)r^2 = Cr(1 + r)^2$.

At the end of year n, the interest earned will be $Cr(1 + r)^{n-1}$.

The total interest earned by the end of year n will be

$$Cr + Cr(1+r) + Cr(1+r)^2 + ... + Cr(1+r)^{n-1}.$$

This is a geometric series of n terms with first term Cr and common ratio $1+r$.

The total interest earned by the end of year n will be

$$\frac{Cr((1+r)^n - 1)}{r} = C((1+r)^n - 1).$$

Note:

The same result can be obtained using only a geometric progression:

At the end of year 1, the total amount will be $C + Cr = C(1+r)$.

At the end of year 2, the total amount will be $C(1+r) + C(1+r)r = C(1+r)^2$.

At the end of year 3, the total amount will be $C(1+r)^2 + C(1+r)^2 r = C(1+r)^3$.

At the end of year n, the total amount will be $C(1+r)^n$.

Total interest earned will be $C(1+r)^n - C = C((1+r)^n - 1)$ as before.

The sum to infinity of a geometric series

In a geometric series with n terms, suppose $|r| < 1$.

As n approaches ∞, r^n approaches 0.

Hence,

$$S_\infty = \frac{a}{1-r}.$$

EXAMPLES

1. Determine the sum to infinity of the geometric series

   $$5 - 1 + \frac{1}{5} - ......$$

   Solution

   The sum to infinity is

   $$\frac{5}{1+\frac{1}{5}} = \frac{25}{6} \simeq 4.17$$

2. The yearly output of a silver mine is found to be decreasing by 25% of its previous year's output. If, in a certain year, its output was £25,000, what could be reckoned as its total future output ?

Solution

The total output, in pounds, for subsequent years will be given by

$$25000 \times 0.75 + 25000 \times (0.75)^2 + 25000 \times (0.75)^3 + ...$$

$$= \frac{25000 \times 0.75}{1 - 0.75} = 75000.$$

2.1.5 MORE GENERAL PROGRESSIONS AND SERIES

Introduction

Not all progressions and series encountered in mathematics are either arithmetic or geometric.

For example

$$1^2, 2^2, 3^2, 4^2, ..., n^2$$

is not arithmetic or geometric.

An **arbitrary** progression of n numbers which conform to some regular pattern is often denoted by

$$u_1, u_2, u_3, u_4, ..., u_n.$$

There may or may not be a simple formula for S_n.

The Sigma Notation (Σ).

1.

$$a+(a+d)+(a+2d)+......+(a+[n-1]d) = \sum_{r=1}^{n} (a+[r-1]d).$$

2.

$$a + ar + ar^2 + ...ar^{n-1} = \sum_{k=1}^{n} ar^{k-1}.$$

3.

$$1^2 + 2^2 + 3^2 + ...n^2 = \sum_{r=1}^{n} r^2.$$

4.

$$-1^3 + 2^3 - 3^3 + 4^3 + ...(-1)^n n^3 = \sum_{r=1}^{n} (-1)^r r^3.$$

Notes:

(i) We sometimes count the terms of a series from zero rather than 1.

For example:

$$a + (a + d) + (a + 2d) + ...a + [n - 1]d = \sum_{r=0}^{n-1} (a + rd).$$

$$a + ar + ar^2 + ar^3 + ...ar^{n-1} = \sum_{k=0}^{n-1} ar^k.$$

In general,

$$u_0 + u_1 + u_2 + u_3 + \ldots\ldots + u_{n-1} = \sum_{r=0}^{n-1} u_r.$$

(ii) We may also use the sigma notation for **"infinite series"**

For example:

$$1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots\ldots = \sum_{r=1}^{\infty} \frac{1}{3^{r-1}} \text{ or } \sum_{r=0}^{\infty} \frac{1}{3^r}.$$

STANDARD RESULTS

It may be shown that

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1),$$

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

and

$$\sum_{r=1}^{n} r^3 = \left[\frac{1}{2}n(n+1)\right]^2.$$

The first of these is the formula for the sum of an arithmetic series with first term 1 and last term n.

The second is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^3 - r^3 \equiv 3r^2 + 3r + 1.$$

The third is proved by summing, from $r = 1$ to n, the identity

$$(r+1)^4 - r^4 \equiv 4r^3 + 6r^2 + 4r + 1.$$

EXAMPLE

Determine the sum to n terms of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + 5 \cdot 6 \cdot 7 + \ldots$$

Solution

The series is

$$\sum_{r=1}^{n} r(r+1)(r+2) = \sum_{r=1}^{n} r^3 + 3r^2 + 2r$$

$$= \sum_{r=1}^{n} r^3 + 3\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r.$$

Using the three standard results, the summation becomes

$$\left[\frac{1}{2}n(n+1)\right]^2 + 3\left[\frac{1}{6}n(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{4}n(n+1)[n(n+1)+4n+2+4] = \frac{1}{4}n(n+1)[n^2+5n+6].$$

This simplifies to

$$\frac{1}{4}n(n+1)(n+2)(n+3).$$