"JUST THE MATHS"

SLIDES NUMBER

19.6

PROBABILITY 6
(Statistics for the binomial distribution)

by

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UNIT 19.6 - PROBABILITY 6

STATISTICS FOR THE BINOMIAL DISTRIBUTION

19.6.1 CONSTRUCTION OF HISTOGRAMS

Frequency tables, histograms etc. usually involve experiments which are actually carried out.

Here, we illustrate how the binomial distribution may be used to estimate the results of a certain kind of experiment before it is performed.

EXAMPLE

For four coins, tossed 32 times, construct a histogram showing the expected number of occurrences of 0,1,2,3,4.....heads.

Solution

Firstly, in a single toss of the four coins, the probability of head (or tail) for each coin is $\frac{1}{2}$.

The terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^4$ give the probabilities of exactly 0,1,2,3 and 4 heads, respectively.

The expansion is

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4 + 4\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + 6\left(\frac{1}{2}\right)^2 + 4\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4.$$

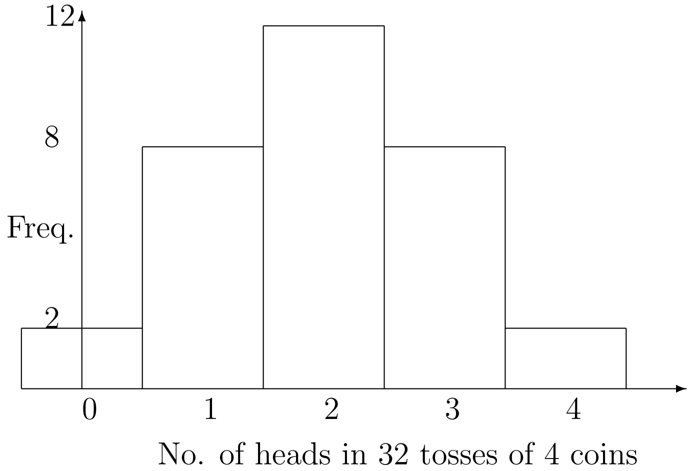
That is,

$$\left(\frac{1}{2}+\frac{1}{2}\right)^4 \equiv \left(\frac{1}{2}\right)^4(1+4+6+4+1).$$

This shows that the probabilities of 0,1,2,3 and 4 heads in a single toss of four coins are $\frac{1}{16}$, $\frac{1}{4}$, $\frac{6}{16}$, $\frac{1}{4}$, and $\frac{1}{16}$, respectively

Therefore, in 32 tosses of four coins, we may expect 0 heads, twice; 1 head, 8 times; 2 heads, 12 times; 3 heads, 8 times and 4 heads, twice.

The following histogram uses class-intervals for which each member is situated at the mid-point:



Notes:

(i) The histogram is symmetrical in shape since the probability of success and failure are equal to each other (the binomial expansion itself is symmetrical).

(ii) Since the widths of the class-intervals in the above histogram are 1, the areas of the rectangles are equal to their heights.

Thus, for example, the total area of the first three rectangles represents the expected number of times of obtaining <u>at most</u> 2 heads in 32 tosses of 4 coins.

19.6.2 MEAN AND STANDARD DEVIATION OF A BINOMIAL DISTRIBUTION

THEOREM

If p is the probability of success of an event in a single trial and q is the probability of its failure, then the binomial distribution, giving the expected frequencies of 0,1,2,3,.....n successes in n trials, has a Mean of np and a Standard Deviation of $\sqrt{npq}$ irrespective of the number of times the experiment is to be carried out.

Proof (Optional):

(a) Mean

From the binomial theorem,

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$\frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ldots + nqp^{n-1} + p^n.$$

Hence, if the n trials are made N times, the average number of successes is equal to the following expression, multiplied by N, then divided by N:

$$0 \times q^n + 1 \times nq^{n-1}p + 2 \times \frac{n(n-1)}{2!}q^{n-2}p^2 +$$

$$3 \times \frac{n(n-1)(n-2)}{3!}q^{n-3}p^3 + \ ..(n-1) \times nqp^{n-1} + np^n.$$

That is,

$$np[q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2}q^{n-3}p^2 + \ \ldots$$

$$+(n-1)qp^{n-2} + p^{n-1}]$$

$$= np(q+p)^{n-1} = np \ \text{ since } \ q+p=1.$$

(b) Standard Deviation

For the standard deviation, we observe that, if f_r is the frequency of r successes when the n trials are conducted N times, then

$$f_r = N\frac{n!}{(n-r)!r!}q^{n-r}p^r.$$

We use this, first, to establish a result for

$$\sum_{r=0}^{n} r^2 f_r.$$

For example,

$0^2 f_0 = 0.Nq^n = 0.f_0$ and $1^2 f_1 = 1.Nnq^{n-1}p = 1.f_1$;

$$2^2 f_2 = 2Nn(n-1)q^{n-2}p^2$$

$$= Nn(n-1)q^{n-2}p^2 + Nn(n-1)p^2q^{n-2}$$

$$= 2f_2 + Nn(n-1)p^2q^{n-2};$$

$$3^2 f_3 = 3N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3$$

$$= N\frac{n(n-1)(n-2)}{2!}q^{n-3}p^3 + Nn(n-1)p^2(n-2)q^{n-3}p$$

$$3^2 f_3 = 3f_3 + Nn(n-1)p^2(n-2)q^{n-3}p;$$

$$4^2 f_4 = 4N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$$

$$= N\frac{n(n-1)(n-2)(n-3)}{3!}q^{n-4}p^4$$

$$+Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2$$

$$= 4f_4 + Nn(n-1)p^2\frac{(n-2)(n-3)}{2!}q^{n-4}p^2.$$

In general, when $r \geq 2$, $r^2 f_r =$

$$N\frac{n(n-1)(n-2)....(n-r+1)}{(r-1)!}+Nn(n-1)p^2q^{n-r}p^r =$$

$$rf_r + Nn(n-1)p^2\frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

$$\sum_{r=0}^{n} r^2 f_r = \sum_{r=0}^{n} rf_r + Nn(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)!}{(n-r)!(r-2)!}q^{n-r}p^{r-2}.$$

Since $q + p = 1$, we have

$$\sum_{r=0}^{n} r^2 f_r = Nnp + Nn(n-1)p^2(q+p)^{n-2}$$

$$= Nnp + Nn(n-1)p^2.$$

The standard deviation of a set $x_1, x_2, x_3, \ldots x_m$ of m observations, with a mean value of $\overline{x}$ is given by the formula

$$\sigma^2 = \frac{1}{m}\sum_{i=1}^{m} x_i^2 - \overline{x}^2.$$

In the present case, this may be written

$$\sigma^2 = \frac{1}{N}\sum_{r=0}^{n} r^2 f_r - \frac{1}{N^2}\left(\sum_{r=0}^{n} r f_r\right)^2.$$

Hence,

$$\sigma^2 = \frac{1}{N}\left(Nnp + Nn(n-1)p^2\right) - \frac{1}{N^2}(Nnp)^2.$$

This gives

$$\sigma^2 = np + n^2p^2 - np^2 - n^2p^2 = np(1-p) = npq.$$

Therefore, $\sigma = \sqrt{npq}$.

ILLUSTRATION

For direct calculation of the mean and the standard deviation for the data in the previous coin-tossing problem, we may use the following table in which x_i denotes numbers of heads and f_i denotes the corresponding expected frequencies:

x_i	f_i	$f_i x_i$	$f_i x_i^2$
0	2	0	0
1	8	8	8
2	12	24	48
3	8	24	72
4	2	8	32
Totals	32	64	160

The mean is given by

$$\overline{x} = \frac{64}{32} = 2 \quad \text{(obviously)}.$$

This agrees with $np = 4 \times \frac{1}{2}$.

The standard deviation is given by

$$\sigma = \sqrt{\left[\frac{160}{32} - 2^2\right]} = 1.$$

This agrees with $\sqrt{npq} = \sqrt{4 \times \frac{1}{2} \times \frac{1}{2}}$.

Note:
If the experiment were carried out N times instead of 32 times, all values in the last three columns of the above table would be multiplied by a factor of $\frac{N}{32}$ which would then cancel out in the remaining calculations.

EXAMPLE

Three dice are rolled 216 times. Construct a binomial distribution and show the frequencies of occurrence for 0,1,2 and 3 sixes.

Evaluate the Mean and the standard deviation of the distribution.

Solution

The probability of success in obtaining a six with a single throw of a die is $\frac{1}{6}$ and the corresponding probability of failure is $\frac{5}{6}$.

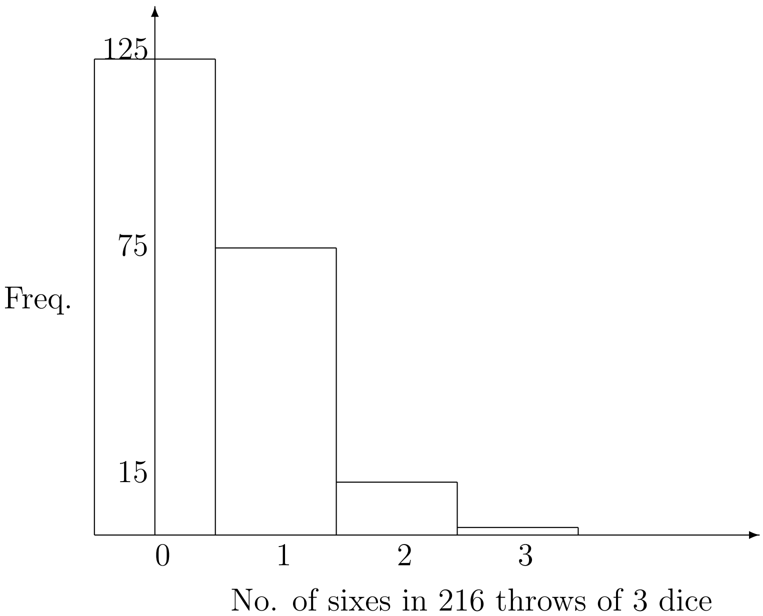
For a single throw of three dice, we require the expansion

$$\left(\frac{1}{6}+\frac{5}{6}\right)^3 \equiv \left(\frac{1}{6}\right)^3 + 3\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right) + 3\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3.$$

This shows that the probabilities of 0,1,2 and 3 sixes are $\frac{125}{216}$, $\frac{75}{216}$, $\frac{15}{216}$ and $\frac{1}{216}$, respectively

Hence, in 216 throws of the three dice we may expect 0 sixes, 125 times; 1 six, 75 times; 2 sixes, 15 times and 3 sixes, once.

The corresponding histogram is as follows:



From the previous Theorem, the mean value is

$$3 \times \frac{1}{6} = \frac{1}{2}$$

and the standard deviation is

$$\sqrt{3 \times \frac{1}{6} \times \frac{5}{6}} = \frac{\sqrt{15}}{6}.$$