"JUST THE MATHS"

SLIDES NUMBER

19.5

PROBABILITY 5
(The binomial distribution)

by

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19.5.1 Introduction and theory

UNIT 19.5 - PROBABILITY 5

THE BINOMIAL DISTRIBUTION

19.5.1 INTRODUCTION AND THEORY

In this Unit, we consider, first, probability problems having only **two** events (mutually exclusive and independent), although many trials may be possible.

For example, the pairs of events could be "up and down", "black and white", "good and bad", and, in general, "successful and unsuccessful".

Statement of the problem

If the probability of success in a single trial is unaffected when successive trials are carried out (independent events), then what is the probability that, in n successive trials, **exactly** r will be successful ?

General Analysis of the problem

We build up the solution in simple stages:

(a) If p is the probability of success in a single trial, then the probability of failure is $1 - p = q$, say.

(b) In the following table, let S stand for success and let F stand for failure:

The table shows the possible results of one, two or three trials and their corresponding probabilities:

Trials	Possible Results	Respective Probabilities
1	F,S	q, p
2	FF,FS,SF,SS	q^2, qp, pq, p^2
3	FFF,FFS,FSF,FSS, SFF,SFS,SSF,SSS	$q^3, q^2p, q^2p, qp^2,$ q^2p, qp^2, qp^2, p^3

(c) **Summary**

(i) In **one** trial, the probabilities that there will be exactly 0 or exactly 1 successes are the respective terms of the expression

$$q + p.$$

(ii) In **two** trials, the probabilities that there will be exactly 0, exactly 1 or exactly 2 successes are the respective terms of the expression

$$q^2 + 2qp + p^2; \quad \text{that is,} \quad (q + p)^2.$$

(iii) In **three** trials, the probabilities that there will be exactly 0, exactly 1, exactly 2 or exactly 3 successes are the respective terms of the expression

$$q^3 + 3q^2p + 3qp^2 + p^3; \quad \text{that is,} \quad (q + p)^3.$$

(iv) In **any number** of trials, n the probabilities that there will be exactly 0, exactly 1, exactly 2, exactly 3, ... or exactly n successes are the respective terms in the binomial expansion of the expression

$$(q+p)^n.$$

(d) **MAIN RESULT:**

The probability that, in n trials, there will be exactly r successes, is the term containing p^r in the binomial expansion of $(q+p)^n$.

It can be shown that this is the value of

$$^nC_r q^{n-r} p^r.$$

EXAMPLES

1. Determine the probability that, in 6 tosses of a coin, there will be exactly 4 heads.

   Solution

   $$q = 0.5, \quad p = 0.5, \quad n = 6, \quad r = 4.$$

   Hence, the required probability is given by

   $$^6C_4.(0.5)^2.(0.5)^4 = \frac{6!}{2!4!}.\frac{1}{4}.\frac{1}{16} = \frac{15}{64}.$$

2. Determine the probability of obtaining the most probable number of heads in 6 tosses of a coin.

   Solution

   The most probable number of heads is given by

$$\frac{1}{2} \times 6 = 3.$$

   The probability of obtaining exactly 3 heads is given by

$${}^{6}C_{3}.(0.5)^{3}(0.5)^{3} = \frac{6!}{3!3!}.\frac{1}{8}.\frac{1}{8} = \frac{20}{64} \simeq 0.31$$

3. Determine the probability of obtaining exactly 2 fives in 7 throws of a die.

   Solution

$$q = \frac{5}{6}, \quad p = \frac{1}{6}, \quad n = 7, \quad r = 2.$$

   Hence, the required probability is given by

$${}^{7}C_{2}.\left(\frac{5}{6}\right)^{5}.\left(\frac{1}{6}\right)^{2} = \frac{7!}{5!2!}.\left(\frac{5}{6}\right)^{5}.\left(\frac{1}{6}\right)^{2} \simeq 0.234$$

4. Determine the probability of throwing at most 2 sixes in 6 throws of a die.

   Solution

   The phrase "at most 2 sixes" means exactly 0, or exactly 1, or exactly 2 sixes.

Hence, we add together the first three terms in the expansion of $(q+p)^6$, where $q=\frac{5}{6}$ and $p=\frac{1}{6}$.

It can be shown that

$$(q+p)^6 = q^6 + 6q^5p + 15q^4p^2 + \ldots\ldots$$

By substituting for q and p, the sum of the first three terms turns out to be

$$\frac{21875}{23328} \simeq 0.938$$

5. It is known that 10% of certain components manufactured are defective. If a random sample of 12 such components is taken, what is the probability that at least 9 are defective ?

   Solution

   The information suggests that removal of components for examination does not affect the probability of 10%.

   This is reasonable since our sample is almost certainly very small compared with all components in existence.

   The probability of success in this exercise is 0.1, even though it refers to defective items, and hence the probability of failure is 0.9

   Using $p = 0.1$, $q = 0.9$, $n = 12$, we require the probabilities (added together) of exactly 9, 10, 11 or 12 defective items.

   These are the last four terms in the expansion of $(q+p)^n$.

That is,

$$^{12}C_9.(0.9)^3.(0.1)^9 + {}^{12}C_{10}.(0.9)^2.(0.1)^{10}$$

$$+^{12}C_{11}.(0.9).(0.1)^{11} + (0.1)^{12}$$

$$\simeq 1.658 \times 10^{-7}.$$

Note:
The use of the **"binomial distribution"** becomes very tedious when the number of trials is large; and two other standard distributions may be used.

They are called the

" normal distribution"

and the

"Poisson distribution".