“JUST THE MATHS”

SLIDES NUMBER

17.8

NUMERICAL MATHEMATICS 8
(Numerical solution)
of
(ordinary differential equations (C))

by

A.J.Hobson

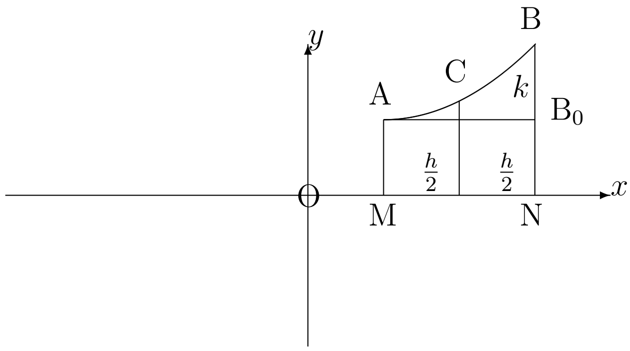
UNIT 17.8
NUMERICAL MATHEMATICS 8

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (C)

17.8.1 Runge's Method

We solve the differential equation, $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$, subject to the condition that $y = y_0$ when $x = x_0$.

Consider the **graph** of the solution passing through the two points, A(x_0, y_0) and B$(x_0 + h, y_0 + k)$.



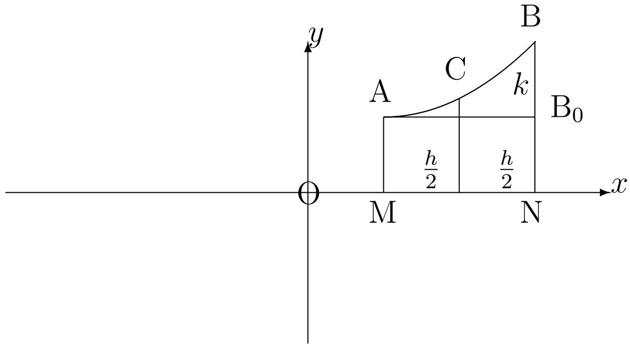
We can say that

$$\int_{x_0}^{x_0+h} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x.$$

That is,

$$y_{\mathrm{B}} - y_{\mathrm{A}} = \int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x.$$

Reminder: $f(x, y)$ is the gradient at points on the solution curve.



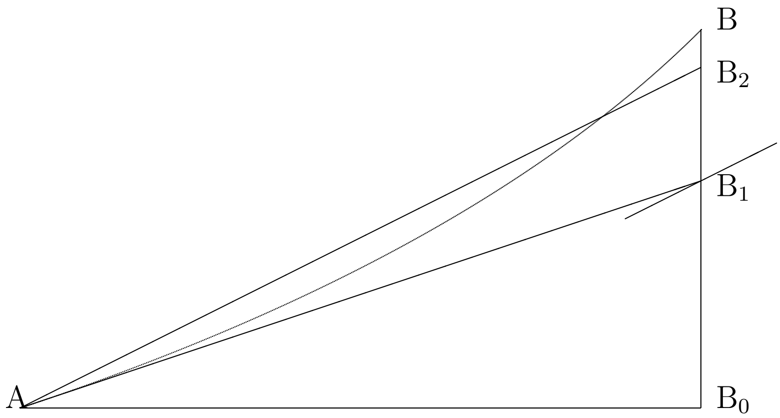
Suppose we knew the values of $f(x, y)$ at A,B and C, where C is the intersection with the curve of the perpendicular bisector of MN.

Then, by Simpson's Rule for approximate integration,

$$\int_{x_0}^{x_0+h} f(x, y)\mathrm{d}x = \frac{h/2}{3}\left[f(\mathrm{A}) + \mathrm{f(B)} + 4\mathrm{f(C)}\right].$$

(i) The value of $f(\mathrm{A})$
This is already given, namely, $f(x_0, y_0)$.

(ii) The Value of $f(\mathrm{B})$



If the tangent at A meets $\mathrm{B}_0\mathrm{B}$ in B_1, then the gradient at A is given by

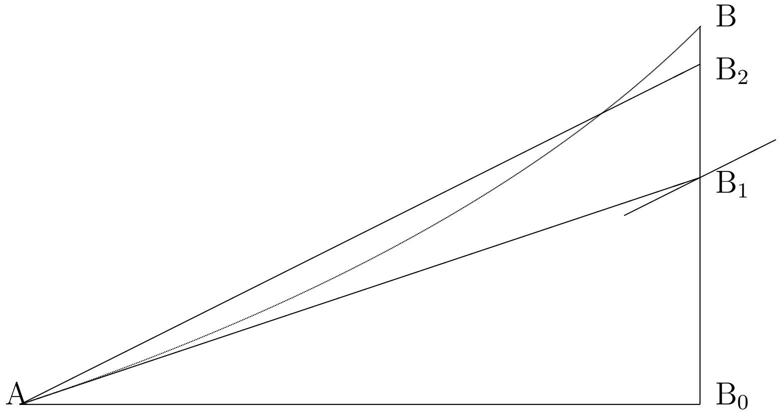
$$\frac{\mathrm{B}_1\mathrm{B}_0}{\mathrm{AB}_0} = f(x_0, y_0).$$

Therefore,

$$\mathrm{B}_1\mathrm{B}_0 = \mathrm{AB}_0 f(x_0, y_0) = hf(x_0, y_0).$$

Calling this value k_1 as an initial approximation to k,

$$k_1 = hf(x_0, y_0).$$



As a rough approximation to the gradient of the solution curve passing through B, we now take the gradient of the solution curve passing through B_1.

Its value is

$$f(x_0 + h, y_0 + k_1).$$

For a better approximation, assume that a straight line of gradient $f(x_0 + h, y_0 + k_1)$, drawn at A, meets B_0B in B_2 a point nearer to B than B_1.

Letting $B_0B_2 = k_2$,

$$k_2 = hf(x_0 + h, y_0 + k_1).$$

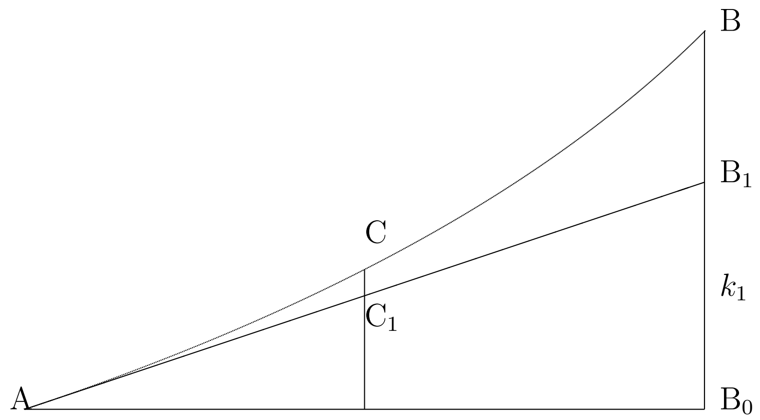
The co-ordinates of B_2 are $(x_0 + h, y_0 + k_2)$.

The gradient of the solution curve through B_2 is taken as a closer approximation than before to the gradient of the solution curve through B.

The gradient of the solution curve through B_2 is

$$f(x_0 + h, y_0 + k_2).$$

(iii) The Value of $f(C)$



Let C_1 be the intersection of the ordinate through C and the tangent at A.

Then C_1 is the point

$$\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

The gradient at C_1 of the solution curve through C_1 is

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right).$$

We take this to be an approximation to the gradient at C for the arc, AB.

We saw earlier that

$$y_{\rm B} - y_{\rm A} = \int_{x_0}^{x_0+h} f(x,y){\rm d}x.$$

Therefore,

$$y_{\rm B} - y_{\rm A} = \frac{h}{6}[f({\rm A}) + {\rm f(B)} + 4{\rm f(C)}].$$

That is, $y =$

$$y_0 + \frac{h}{6}\left[f(x_0,y_0) + f(x_0+h, y_0+k_2) + 4f\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right)\right].$$

PRACTICAL LAYOUT

If

$$\frac{{\rm d}y}{{\rm d}x} = f(x,y)$$

and $y = y_0$ when $x = x_0$, then the value of y when $x = x_0 + h$ is determined by the following sequence of calculations:

1. $k_1 = hf(x_0, y_0)$.
2. $k_2 = hf(x_0 + h, y_0 + k_1)$.
3. $k_3 = hf(x_0 + h, y_0 + k_2)$.
4. $k_4 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$.
5. $k = \frac{1}{6}(k_1 + k_3 + 4k_4)$.
6. $y \simeq y_0 + k$.

EXAMPLE

Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 5 - 3y$$

at $x = 0.1$ given that $y = 1$ when $x = 0$

Solution

We use $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

1. $k_1 = 0.1(5 - 3) = 0.2$
2. $k_2 = 0.1(5 - 3[1.2]) = 0.14$
3. $k_3 = 0.1(5 - 3[1.14]) = 0.158$
4. $k_4 = 0.1(5 - 3[1.1]) = 0.17$
5. $k = \frac{1}{6}(0.2 + 0.158 + 4[0.17]) = 0.173$
6. $y \simeq 1.173$ at $x = 0.1$

Note: It can be shown that the error in the result is of the order h^5; that is, the error is equivalent to some constant multiplied by h^5.