“JUST THE MATHS”

SLIDES NUMBER

17.7

NUMERICAL MATHEMATICS 7
(Numerical solution)
of
(ordinary differential equations (B))

by

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17.7.1 Picard’s method

UNIT 17.7
NUMERICAL MATHEMATICS 7

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (B)

17.7.1 PICARD'S METHOD

This method of solution is one of successive approximation. It is an **iterative** method in which the numerical results become more and more accurate, the more times it is used.

An approximation for y in terms of x is substituted into the right hand side of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

The equation is then integrated with respect to x, giving y in terms of x as a second approximation.

Given numerical values are substituted into the second approximation and the result rounded off to an assigned number of decimal places or significant figures.

The iterative process is continued until two consecutive numerical solutions are the same when rounded off to the required number of decimal places.

A hint on notation

Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2,$$

given that $y = y_0 = 7$ when $x = x_0 = 2$.

This <u>can</u> be solved exactly to give

$$y = x^3 + C,$$

which requires that

$$7 = 2^3 + C.$$

Hence,

$$y - 7 = x^3 - 2^3;$$

or, in more general terms,

$$y - y_0 = x^3 - x_0^3.$$

Thus,

$$\int_{y_0}^{y} \mathrm{d}y = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

That is,

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x} \mathrm{d}x = \int_{x_0}^{x} 3x^2 \mathrm{d}x.$$

In future, we shall integrate both sides of the given differential equation with respect to x, from x_0 to x.

EXAMPLES

1. Given that
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y^2,$$
and that $y = 0$ when $x = 0$, determine the value of y when $x = 0.3$, correct to four places of decimals.

Solution

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} (x + y^2)\mathrm{d}x,$$

where $x_0 = 0$

Hence,
$$y - y_0 = \int_{x_0}^{x} (x + y^2)\mathrm{d}x,$$
where $y_0 = 0$.

That is,
$$y = \int_0^x (x + y^2)\mathrm{d}x.$$

(a) First Iteration

Replace y by y_0 in the function to be integrated.

$$y_1 = \int_0^x x\mathrm{d}x = \frac{x^2}{2} \simeq 0.0450 \quad \text{at} \quad x = 0.3$$

(b) Second Iteration

Now we use
$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_1^2 = x + \frac{x^4}{4}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20}.$$

$$y_2 = \frac{x^2}{2} + \frac{x^5}{20} \simeq 0.0451 \quad \text{at} \quad x = 0.3$$

(c) Third Iteration

Now we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x + y_2^2$$
$$= x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}.$$

Therefore,

$$\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_0^x \left(x + \frac{x^4}{4} + \frac{x^7}{20} + \frac{x^{10}}{400}\right)\mathrm{d}x,$$

which gives

$$y - 0 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400}.$$

$$y_3 = \frac{x^2}{2} + \frac{x^5}{20} + \frac{x^8}{160} + \frac{x^{11}}{4400} \simeq 0.0451 \quad \text{at} \quad x = 0.3$$

2. If

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y}{x},$$

and $y = 2$ when $x = 1$, perform three iterations of Picard's method to estimate a value for y when $x = 1.2$. Work to four places of decimals throughout and state how accurate is the result of the third iteration.

(a) First Iteration

$$\int_{x_0}^{x} \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $x_0 = 1$.

That is,

$$y - y_0 = \int_{x_0}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x,$$

where $y_0 = 2$.

Hence,

$$y - 2 = \int_{1}^{x} \left(2 - \frac{y}{x}\right) \mathrm{d}x.$$

Replacing y by $y_0 = 2$ in the function being integrated, we have

$$y - 2 = \int_{1}^{x} \left(2 - \frac{2}{x}\right) \mathrm{d}x.$$

Therefore,

$$y = 2 + [2x - 2\ln x]_1^x$$

$$= 2 + 2x - 2\ln x - 2 + 2\ln 1 = 2(x - \ln x).$$

$$y_1 = 2(x - \ln x) \simeq 2.0354 \quad \text{when} \quad x = 1.2$$

(b) Second Iteration

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_1}{x} = 2 - \frac{2(x - \ln x)}{x} = \frac{2\ln x}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \frac{2lnx}{x}\mathrm{d}x.$$

That is,

$$y - 2 = \left[(\ln x)^2\right]_1^x = (\ln x)^2.$$

$$y_2 = 2 + (\ln x)^2 \simeq 2.0332 \quad \text{when} \quad x = 1.2$$

(c) Third Iteration

Finally, we use

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - \frac{y_2}{x} = 2 - \frac{2}{x} - \frac{(\ln x)^2}{x}.$$

Hence,

$$\int_1^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = \int_1^x \left[2 - \frac{2}{x} - \frac{(lnx)^2}{x}\right]\mathrm{d}x.$$

That is,

$$y - 2 = \left[2x - 2\ln x - \frac{(\ln x)^3}{3}\right]_1^x.$$

$$y - 2 = 2x - 2\ln x - \frac{(\ln x)^3}{3} - 2.$$

$$y_3 = 2x - 2\ln x - \frac{(\ln x)^3}{3} \simeq 2.0293 \quad \text{when} \quad x = 1.2$$

The results of the last two iterations are identical when rounded off to two places of decimals, namely 2.03.