"JUST THE MATHS"

SLIDES NUMBER

17.6

NUMERICAL MATHEMATICS 6 (Numerical solution) of (ordinary differential equations (A))

by

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UNIT 17.6
NUMERICAL MATHEMATICS 6

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS (A)

17.6.1 EULER'S UNMODIFIED METHOD

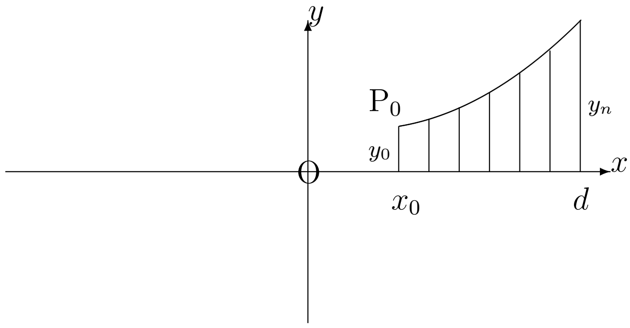
Every first order ordinary differential equation can be written in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y).$$

If $y = y_0$ when $x = x_0$, then the solution for y in terms of x represents some curve through the point $\mathrm{P}_0(x_0, y_0)$.

Suppose we need y at $x = d$, where $d > x_0$.

Sub-divide the interval from $x = x_0$ to $x = d$ into n equal parts of width δx.



Let $x_1, x_2, x_3, ...$ be the points of subdivision.

$$x_1 = x_0 + \delta x,$$
$$x_2 = x_0 + 2\delta x,$$
$$x_3 = x_0 + 3\delta x,$$
$$...,$$
$$...,$$
$$d = x_n = x_0 + n\delta x.$$

If $y_1, y_2, y_3, ...$ are the y co-ordinates of $x_1, x_2, x_3, ...,$ we must find y_n.

The increase in y, when x increases by $\delta x \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x$.

Since $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$,

$$
\begin{aligned}
y_1 &= y_0 + f(x_0, y_0)\delta x, \\
y_2 &= y_1 + f(x_1, y_1)\delta x, \\
y_3 &= y_2 + f(x_2, y_2)\delta x, \\
&\dots, \\
&\dots, \\
y_n &= y_{n-1} + f(x_{n-1}, y_{n-1})\delta x.
\end{aligned}
$$

Each stage uses the previously calculated y value.

Note:
The method will be the same if $d < x_0$ except that δx will be negative.

In general,

$$y_{i+1} = y_i + f(x_i, y_i)\delta x.$$

EXAMPLE

Use Euler's method with 5 sub-intervals to continue to $x = 0.5$ the solution of the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

i	x_i	y_i	$f(x_i, y_i)$	$y_{i+1} = y_i + f(x_i, y_i)\delta x$
0	0	1	0	1
1	0.1	1	0.1	1.01
2	0.2	1.01	0.202	1.0302
3	0.3	1.0302	0.30906	1.061106
4	0.4	1.061106	0.4244424	1.1035524
5	0.5	1.1035524	-	-

Accuracy

Here, we may compare the exact result with the approximation by Eulers' method.

$$\int \frac{\mathrm{d}y}{y} = \int x\mathrm{d}x.$$

$$\ln y = \frac{x^2}{2} + C.$$

$$y = Ae^{\frac{x^2}{2}}.$$

At $x = 0$, $y = 1$ and, hence, $A = 1$.

$$y = e^{\frac{x^2}{2}}.$$

But a table of values of x against y reveals the following:

x	$e^{\frac{x^2}{2}}$
0	1
0.1	1.00501
0.2	1.0202
0.3	1.04603
0.4	1.08329
0.5	1.13315

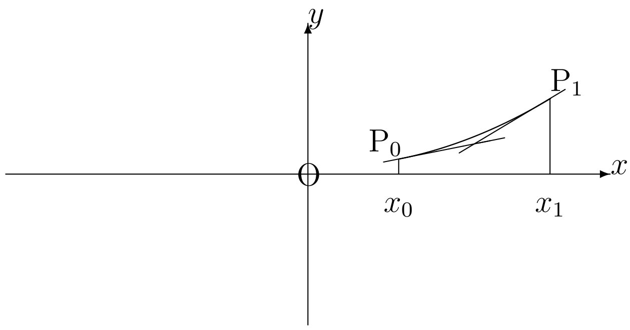
There is an error in our approximate value of 0.0296, which is about 2.6%.

Attempts to determine y for values of x which are greater than 0.5 would result in a very rapid growth of error.

17.6.2 EULER'S MODIFIED METHOD

In the previous method, we used the gradient at P_0 in order to find P_1, and so on up to P_n.

But the approximation is better if we use the **average** of the two gradients at P_0 and P_1.



The gradient, m_0 at P_0 is given by

$$m_0 = f(x_0, y_0).$$

The gradient, m_1 at P_1 is given approximately by

$$m_1 = f(x_0 + \delta x, y_0 + \delta y_0),$$

where $\delta y_0 = f(x_0, y_0)\delta x$.

The average gradient between P_0 and P_1 is given by

$$m_0^* = \frac{1}{2}(m_0 + m_1).$$

The modified approximation to y at the point P_1 is given by

$$y_1 = y_0 + m_0^* \delta x.$$

Similarly, we proceed from y_1 to y_2 and so on until we reach y_n.

In general,

$$y_{i+1} = y_i + m_i^* \delta x.$$

EXAMPLE

Use Euler's modified method with 5 sub-intervals to continue to $x = 0.5$ the solution to the differential equation,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = xy,$$

given that $y = 1$ when $x = 0$; (that is, $y(0) = 1$).

i	x_i	y_i	$m_i = f(x_i, y_i)$	$\delta y_i = f(x_i, y_i)\delta x$
0	0	1	0	0
1	0.1	1.005	0.1005	0.0101
2	0.2	1.0202	0.2040	0.0204
3	0.3	1.0460	0.3138	0.0314
4	0.4	1.0832	0.4333	0.0433
5	0.5	1.1330	——	——

i	$m_{i+1} = f(x_i + \delta x, y_i + \delta y_i$	$m_i^* = \frac{1}{2}(m_i + m_{i+1})$	$y_{i+1} = y_i + m_i^*\delta x$
0	0.1	0.05	1.005
1	0.2030	0.1518	1.0202
2	0.3122	0.2581	1.0460
3	0.4310	0.3724	1.0832
4	0.5633	0.4983	1.1330
5	——	——	——