"JUST THE MATHS"

SLIDES NUMBER

17.5

NUMERICAL MATHEMATICS 5
(Iterative methods)
for solving
(simultaneous linear equations)

by

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UNIT 17.5 - NUMERICAL MATHEMATICS 5

ITERATIVE METHODS FOR SOLVING SIMULTANEOUS LINEAR EQUATIONS

17.5.1 INTRODUCTION

An **"iterative method"** is one which is used repeatedly until the results obtained acquire a pre-assigned degree of accuracy.

For example, if results are required to five places of decimals, the number of **"iterations"** is continued until two consecutive iterations give the same result when rounded off to that number of decimal places.

It is usually enough for the calculations themselves to be carried out to **two extra** places of decimals.

A similar interpretation holds for accuracy which requires a certain number of **significant figures**.

We shall discuss sets of simultaneous linear equations of the form

$$\begin{aligned} a_1x + b_1y + c_1z &= k_1, \\ a_2x + b_2y + c_2z &= k_2, \\ a_3x + b_3y + c_3z &= k_3. \end{aligned}$$

The system must be **"diagonally dominant"**, which, in this case, means that

$$\begin{aligned} |a_1| &> |b_1| + |c_1|, \\ |b_2| &> |a_2| + |c_2|, \\ |c_3| &> |a_3| + |b_3|. \end{aligned}$$

The methods would be adaptable to a different number of simultaneous equations.

17.5.2 THE GAUSS-JACOBI ITERATION

This method begins by making x the subject of the first equation, y the subject of the second equation and z the subject of the third equation.

An initial approximation $x_0 = 1, y_0 = 1, z_0 = 1$ is substituted on the new right-hand sides to give values $x = x_1, y = y_1$ and $z = z_1$ on the new left-hand sides.

The results of the $(n+1)$-th iteration are as follows:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_n \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_n - b_3 y_n \qquad\right).
\end{aligned}
$$

EXAMPLES

1. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$
\begin{aligned}
5x + y - z &= 4, \\
x + 4y + 2z &= 15, \\
x - 2y + 5z &= 12,
\end{aligned}
$$

obtaining x, y and z correct to the nearest integer.

Solution

$$\begin{aligned} x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\ y_{n+1} &= 3.75 - 0.25x_n \qquad - 0.5z_n, \\ z_{n+1} &= 2.4 - 0.2x_n + 0.4y_n. \end{aligned}$$

Using

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1,$$

we obtain

$$x_1 = 0.8, \quad y_1 = 3.0, \quad z_1 = 2.6,$$

$$x_2 = 0.72, \quad y_2 = 2.25, \quad z_2 = 3.44,$$

$$x_3 = 1.038, \quad y_3 = 1.85, \quad z_3 = 3.156$$

The results of the last two iterations both give

$$x = 1, \quad y = 2, \quad z = 3,$$

when rounded to the nearest integer.

In fact, these whole numbers are clearly seen to be the **exact** solutions.

2. Use the Gauss-Jacobi method to solve the simultaneous linear equations

$$\begin{aligned} x + 7y - z &= 3, \\ 5x + y + z &= 9, \\ -3x + 2y + 7z &= 17, \end{aligned}$$

obtaining x, y and z correct to the nearest integer.

Solution

This set of equations is not diagonally dominant; but they can be rewritten as

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17. \end{aligned}$$

Note:

We could also interchange the first two of the original equations.

Thus,

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_n - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_n - 0.29y_n. \end{aligned}$$

Using

$$y_0 = 1, \quad x_0 = 1, \quad z_0 = 1,$$

we obtain

$$y_1 = 0.43, \quad x_1 = 1.4, \quad z_1 = 2.57,$$

$$y_2 = 0.59, \quad x_2 = 1.2, \quad z_2 = 2.91,$$

$$y_3 = 0.67, \quad x_3 = 1.1, \quad z_3 = 2.78$$

Hence, $x = 1, \quad y = 1, \quad z = 3$ to the nearest integer.

17.5.3 THE GAUSS-SEIDEL ITERATION

This method differs from the Gauss-Jacobi Iteration in that successive approximations are used within each step **as soon as they become available**.

The rate of convergence of this method is usually faster than that of the Gauss-Jacobi method.

The scheme of the calculations is according to the following pattern:

$$
\begin{aligned}
x_{n+1} &= \frac{1}{a_1}\left(k_1 \qquad - b_1 y_n - c_1 z_n\right), \\
y_{n+1} &= \frac{1}{b_2}\left(k_2 - a_2 x_{n+1} \qquad - c_2 z_n\right), \\
z_{n+1} &= \frac{1}{c_3}\left(k_3 - a_3 x_{n+1} - b_3 y_{n+1} \qquad\right).
\end{aligned}
$$

EXAMPLES

1. Use the Gauss-Seidel method to solve the simultaneous linear equations

$$\begin{aligned} 5x + y - z &= 4, \\ x + 4y + 2z &= 15, \\ x - 2y + 5z &= 12. \end{aligned}$$

Solution

$$\begin{aligned} x_{n+1} &= 0.8 \qquad - 0.2y_n + 0.2z_n, \\ y_{n+1} &= 3.75 - 0.25x_{n+1} \qquad - 0.5z_n, \\ z_{n+1} &= 2.4 - 0.2x_{n+1} + 0.4y_{n+1}. \end{aligned}$$

The sequence of successive results is as follows:

$$x_0 = 1, \quad y_0 = 1, \quad z_0 = 1,$$

$$x_1 = 0.8, \quad y_1 = 3.05, \quad z_1 = 3.46,$$

$$x_2 = 0.88, \quad y_2 = 1.80, \quad z_2 = 2.94,$$

$$x_3 = 1.03, \quad y_3 = 2.02, \quad z_3 = 3.00$$

Hence, $x = 1, \quad y = 2, \quad z = 3$ to the nearest integer.

2. Use the Gauss-Seidel method to solve the simultaneous linear equations:

$$\begin{aligned} 7y + x - z &= 3, \\ y + 5x + z &= 9, \\ 2y - 3x + 7z &= 17. \end{aligned}$$

Solution

$$\begin{aligned} y_{n+1} &= 0.43 - 0.14x_n \qquad + 0.14z_n, \\ x_{n+1} &= 1.8 \qquad - 0.2y_{n+1} - 0.2z_n, \\ z_{n+1} &= 2.43 + 0.43x_{n+1} - 0.29y_{n+1}. \end{aligned}$$

The sequence of successive results is:

$$y_0 = 1, \quad x_0 = 1, \quad z_0 = 1,$$

$$y_1 = 0.43, \quad x_1 = 1.51, \quad z_1 = 2.96,$$

$$y_2 = 0.63, \quad x_2 = 1.08, \quad z_2 = 2.71,$$

$$y_3 = 0.66, \quad x_3 = 1.13, \quad z_3 = 2.73$$

The solutions are $x = 1, y = 1, z = 3$ to the nearest integer.