

**“JUST THE MATHS”**

**SLIDES NUMBER**

**17.1**

**NUMERICAL MATHEMATICS 1  
(Approximate solution of equations)**

**by**

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## UNIT 17.1

### NUMERICAL MATHEMATICS 1

## THE APPROXIMATE SOLUTION OF ALGEBRAIC EQUATIONS

### 17.1.1 INTRODUCTION

In the work which follows, we shall consider the solution of the equation  $f(x) = 0$ , where  $f(x)$  is a given function of  $x$ .

The equation  $f(x) = 0$  cannot, in general, be solved algebraically to give **exact** solutions.

It is often possible to find approximate solutions which are correct to any specified degree of accuracy.

**Graphical** methods of solving the equation,  $f(x) = 0$ , use a graph of the equation  $y = f(x)$  to determine where the graph crosses the  $x$ -axis.

“**Iterative**” methods involve **repeated** use of a technique to improve the accuracy of an approximate solution already obtained.

## 17.1.2 THE BISECTION METHOD

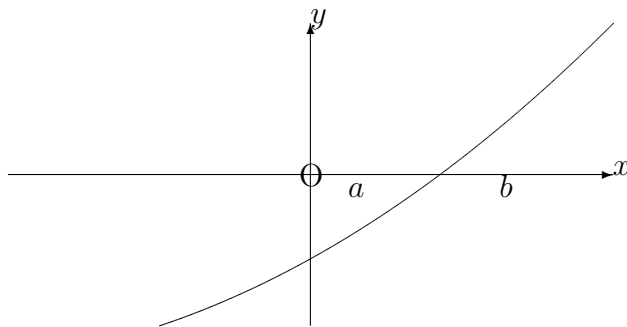
Suppose  $a$  and  $b$  are two numbers such that  $f(a) < 0$  and  $f(b) > 0$ .

We may obtain these,

(a) by trial and error;

(b) by sketching, roughly, the graph of the equation

$$y = f(x).$$



Whole number values of  $a$  and  $b$  will usually suffice.

If we let  $c = (a + b)/2$ , there are three possibilities;

(i)  $f(c) = 0$ , in which case we have solved the equation;

(ii)  $f(c) < 0$ , in which case there is a solution between  $b$  and  $c$ ; so repeat the procedure with  $b$  and  $c$ ;

(iii)  $f(c) > 0$ , in which case there is a solution between  $a$  and  $c$ ; so repeat the procedure with  $a$  and  $c$ .

Each time we apply the method, we bisect the interval between the two numbers being used.

Eventually, the two numbers used will be very close together.

The method stops when two consecutive values of the mid-point agree with each other to the required number of decimal places or significant figures.

Convenient labels for the three numbers used at each stage (or iteration) are

$$a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2, a_3, b_3, c_3, \dots, a_n, b_n, c_n, \dots$$

## EXAMPLE

Determine, correct to three decimal places, the positive solution of the equation

$$e^x = x + 2.$$

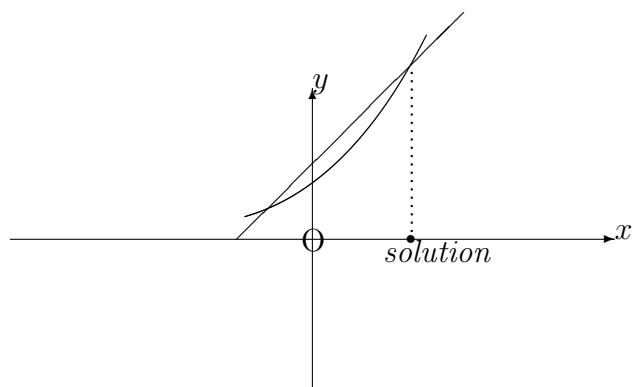
## Solution

The graphs of

$$y = e^x \quad \text{and} \quad y = x + 2$$

intersect each other at a positive value of  $x$ .

This confirms that there is a positive solution to our equation.



But now let

$$f(x) = e^x - x - 2$$

and look for two numbers between which  $f(x)$  changes sign, from positive to negative.

By trial and error, suitable numbers are 1 and 2, since

$$f(1) = e - 3 < 0 \quad \text{and} \quad f(2) = e^2 - 5 > 0.$$

The rest of the solution may be set out in the form of a table as follows:

| $n$ | $a_n$   | $b_n$   | $c_n$   | $f(c_n)$  |
|-----|---------|---------|---------|-----------|
| 0   | 1.00000 | 2.00000 | 1.50000 | 0.98169   |
| 1   | 1.00000 | 1.50000 | 1.25000 | 0.24034   |
| 2   | 1.00000 | 1.25000 | 1.12500 | - 0.04478 |
| 3   | 1.12500 | 1.25000 | 1.18750 | 0.09137   |
| 4   | 1.12500 | 1.18750 | 1.15625 | 0.02174   |
| 5   | 1.12500 | 1.15625 | 1.14062 | - 0.01191 |
| 6   | 1.14063 | 1.15625 | 1.14844 | 0.00483   |
| 7   | 1.14063 | 1.14844 | 1.14454 | - 0.00354 |
| 8   | 1.14454 | 1.14844 | 1.14649 | 0.00064   |
| 9   | 1.14454 | 1.14649 | 1.14552 | - 0.00144 |

As a general rule, it is appropriate to work to two more places of decimals than that of the required accuracy and so, in this case, we work to five.

We can stop at stage 9 since  $c_8$  and  $c_9$  are the same value when rounded to three places of decimals.

The required solution is therefore  $x = 1.146$

### 17.1.3 THE RULE OF FALSE POSITION

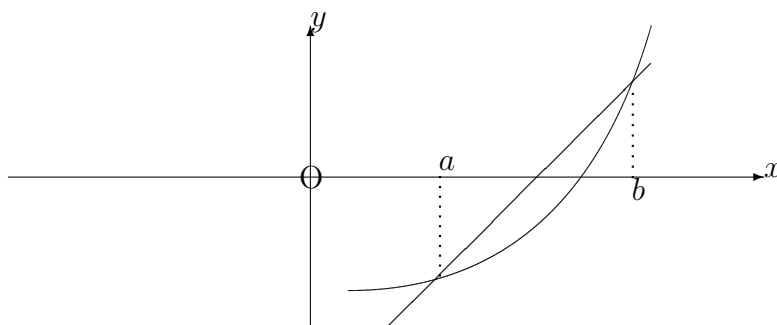
This method is commonly known by its Latin name “**Regula Falsi**”.

We consider that the two points,  $(a, f(a))$  and  $(b, f(b))$ , on the graph of

$$y = f(x)$$

are joined by a straight line.

The point at which this straight line crosses the  $x$ -axis is taken as  $c$ .



From elementary co-ordinate geometry, the equation of the straight line is given by

$$\frac{y - f(a)}{f(b) - f(a)} = \frac{x - a}{b - a}.$$

Hence, when  $y = 0$ , we obtain

$$x = a - \frac{(b - a)f(a)}{f(b) - f(a)}.$$

That is,

$$x = \frac{a[f(b) - f(a)] - (b - a)f(a)}{f(b) - f(a)}.$$

Hence,

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

In setting out the tabular form of a Regula Falsi solution, the  $c_n$  column uses the general formula

$$c_n = \frac{a_n f(b_n) - b_n f(a_n)}{f(b_n) - f(a_n)}.$$

## EXAMPLE

For the equation

$$f(x) \equiv x^3 + 2x - 1 = 0,$$

use the Regula Falsi method, with  $a_0 = 0$  and  $b_0 = 1$ , to obtain the first approximation,  $c_0$ , to the solution between  $x = 0$  and  $x = 1$ .

### Solution

We have  $f(0) = -1$  and  $f(1) = 2$ , so that there is a solution between  $x = 0$  and  $x = 1$ .

From the general formula,

$$c_0 = \frac{0 \times 2 - 1 \times (-1)}{2 - (-1)} = \frac{1}{3}.$$



Continuing with the method, we would observe that  $f(1/3) < 0$  so that  $a_1 = 1/3$  and  $b_1 = 1$ .

**Note:**

The Bisection Method would've given  $c_0 = \frac{1}{2}$ .

### 17.1.4 THE NEWTON-RAPHSON METHOD

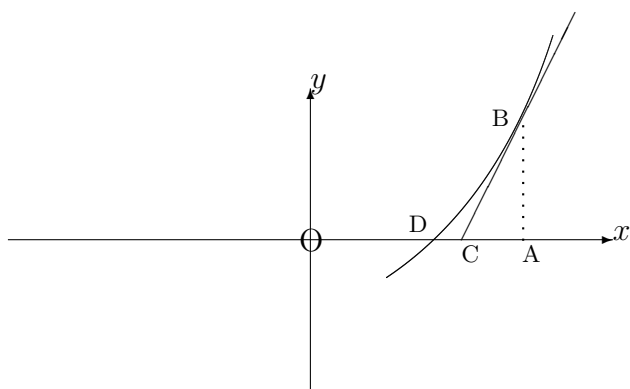
This method is based on first guessing an approximate solution  $x = x_0$  to the equation  $f(x) = 0$ .

We then draw the tangent to the curve, whose equation is

$$y = f(x),$$

at the point,  $x_0, f(x_0)$ , to find out where this tangent crosses the  $x$ -axis.

The point obtained is normally a better approximation,  $x_1$ , to the solution.



In the diagram,

$$f'(x_0) = \frac{AB}{AC} = \frac{f(x_0)}{h}.$$

Hence,

$$h = \frac{f(x_0)}{f'(x_0)}.$$

Thus, a better approximation to the exact solution at the point,  $D$ , is given by

$$x_1 = x_0 - h.$$

Repeating the process, gives rise to the following iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

### **Notes:**

(i) To guess the starting approximation,  $x_0$ , it is normally sufficient to find a pair of whole numbers,  $a$  and  $b$ , such that  $f(a) < 0$  and  $f(b) > 0$ ; then we take  $x_0 = (a + b)/2$ .

In some exercises, an alternative starting approximation may be suggested in order to speed up the rate of convergence to the final solution.

(ii) There are situations where the Newton-Raphson Method fails to give a better approximation:

When the tangent to the curve has a very small gradient, it meets the  $x$ -axis at a relatively great distance from the previous approximation.

### **EXAMPLE**

Use the Newton-Raphson method to calculate  $\sqrt{5}$  correct to three places of decimals.

### **Solution**

We are required to solve the equation

$$f(x) \equiv x^2 - 5 = 0.$$

By trial and error, a solution exists between  $x = 2$  and  $x = 3$ , since  $f(2) = -1 < 0$  and  $f(3) = 4 > 0$ .

Hence, we use  $x_0 = 2.5$

Furthermore,

$$f'(x) = 2x,$$

so that

$$x_{n+1} = x_n - \frac{x_n^2 - 5}{2x_n}.$$

Thus,

$$\begin{aligned}x_1 &= 2.5 - \frac{1.25}{5} = 2.250, \\x_2 &= 2.250 - \frac{0.0625}{4.5} \simeq 2.236, \\x_3 &= 2.236 - \frac{-0.000304}{4.472} \simeq 2.236\end{aligned}$$

At each stage, we round off the result to the required number of decimal places and use the rounded figure in the next iteration.

The last two iterations give the same result to three places of decimals and this is therefore the required result.