"JUST THE MATHS"

SLIDES NUMBER

16.8

Z-TRANSFORMS 1
(Definition and rules)

by

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UNIT 16.8 - Z TRANSFORMS 1

DEFINITION AND RULES

16.8.1 INTRODUCTION

Linear Difference Equations

We consider "linear difference equations with constant coefficients".

DEFINITION 1

A first-order linear difference equation with constant coefficients has the general form,

$$a_1 u_{n+1} + a_0 u_n = f(n);$$

a_0, a_1 are constants;

n is a positive integer;

$f(n)$ is a given function of n (possibly zero);

u_n is the general term of an infinite sequence of numbers,

$$\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots$$

DEFINITION 2

A second order linear difference equation with constant coefficients has the general form

$$a_2 u_{n+2} + a_1 u_{n+1} + a_0 u_n = f(n);$$

a_0, a_1, a_2 are constants;

n is an integer;

$f(n)$ is a given function of n (possibly zero);

u_n is the general term of an infinite sequence of numbers,

$$\{u_n\} \equiv u_0,\ u_1,\ u_2\ , u_3,\ \ldots$$

Notes:

(i) We shall assume that $u_n = 0$ whenever $n < 0$.

(ii) **"Boundary conditions"** will be given as follows:

The value of u_0 for a first-order equation;

The values of u_0 and u_1 for a second-order equation.

ILLUSTRATION

Certain simple difference equations may be solved by very elementary methods.

For example, to solve

$$u_{n+1} - (n+1)u_n = 0,$$

subject to the boundary condition that $u_0 = 1$,

we may rewrite the difference equation as

$$u_{n+1} = (n+1)u_n.$$

By using this formula repeatedly, we obtain

$$u_1 = u_0 = 1, \quad u_2 = 2u_1 = 2,$$

$$u_3 = 3u_2 = 3 \times 2, \quad u_4 = 4u_3 = 4 \times 3 \times 2, \quad . \quad . \quad . \quad .$$

Hence,

$$u_n = n!$$

16.8.2 STANDARD Z-TRANSFORM DEFINITION AND RESULTS

THE DEFINITION OF A Z-TRANSFORM (WITH EXAMPLES)

The Z-Transform of the sequence of numbers $\{u_n\} \equiv u_0,\ u_1,\ u_2,\ u_3,\ \ldots.$ is defined by the formula,

$$Z\{u_n\} = \sum_{r=0}^{\infty} u_r z^{-r},$$

provided that the series converges (allowing for z to be a complex number if necessary).

EXAMPLES

1. Determine the Z-Transform of the sequence,

$$\{u_n\} \equiv \{a^n\},$$

where a is a non-zero constant.

Solution

$$Z\{a^n\} = \sum_{r=0}^{\infty} a^r z^{-r}.$$

That is,

$$Z\{a^n\} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ \ldots. = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a},$$

by properties of infinite geometric series.

2. Determine the Z-Transform of the sequence,

$$\{u_n\} = \{n\}.$$

Solution

$$Z\{n\} = \sum_{r=0}^{\infty} rz^{-r}.$$

That is,

$$Z\{n\} = \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} + \ \ldots .$$

or

$$\left(\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ \ldots .\right) + \left(\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \ \ldots .\right) +$$

$$\left(\frac{1}{z^3} + \frac{1}{z^4} + \ \ldots .\right),$$

giving

$$Z\{n\} = \frac{\frac{1}{z}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^2}}{1 - \frac{1}{z}} + \frac{\frac{1}{z^3}}{1 - \frac{1}{z}} + \ldots . = \frac{1}{1 - \frac{1}{z}}\left[\frac{\frac{1}{z}}{1 - \frac{1}{z}}\right],$$

by properties of infinite geometric series.

Thus,

$$Z\{n\} = \frac{z}{(1 - z)^2} = \frac{z}{(z - 1)^2}$$

A SHORT TABLE OF Z-TRANSFORMS

$\{u_n\}$	$Z\{u_n\}$	Region
$\{1\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{a^n\}$ (a constant)	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\{n\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{e^{-nT}\}$ (T constant)	$\frac{z}{z-e^{-T}}$	$\lvert z\rvert > e^{-T}$
$\sin nT$ (T constant)	$\frac{z\sin T}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
$\cos nT$ (T constant)	$\frac{z(z-\cos T)}{z^2-2z\cos T+1}$	$\lvert z\rvert > 1$
1 for $n = 0$ 0 for $n > 0$ (Unit Pulse sequence)	1	All z
0 for $n = 0$ $\{a^{n-1}\}$ for $n > 0$	$\frac{1}{z-a}$	$\lvert z\rvert > \lvert a\rvert$

16.8.3 PROPERTIES OF Z-TRANSFORMS

(a) Linearity

If $\{u_n\}$ and $\{v_n\}$ are sequences of numbers, while A and B are constants, then

$$Z\{Au_n + Bv_n\} \equiv A.Z\{u_n\} + B.Z\{v_n\}.$$

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} (Au_r + Bv_r)z^{-r} \equiv A \sum_{r=0}^{\infty} u_r z^{-r} + B \sum_{r=0}^{\infty} v_r z^{-r}.$$

This is equivalent to the right-hand side.

EXAMPLE

$$Z\{5.2^n - 3n\} = \frac{5z}{z-2} - \frac{3z}{(z-1)^2}.$$

(b) The First Shifting Theorem

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.Z\{u_n\},$$

where $\{u_{n-1}\}$ denotes the sequence whose first term, corresponding to $n = 0$, is taken as zero and whose subsequent terms, corresponding to $n = 1,\ 2,\ 3,\ 4,\ \ldots.$, are the terms $u_0,\ u_1,\ u_3,\ u_4,\ \ \ldots$ of the original sequence.

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} u_{r-1}z^{-r} \equiv \frac{u_0}{z} + \frac{u_1}{z^2} + \frac{u_2}{z^3} + \frac{u_3}{z^4} + \ \ldots.,$$

since $u_n = 0$ whenever $n < 0$.

Thus,

$$Z\{u_{n-1}\} \equiv \frac{1}{z}.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \ \ldots.\right].$$

This is equivalent to the right-hand side.

Note:
A more general form of the First Shifting Theorem states that

$$Z\{u_{n-k}\} \equiv \frac{1}{z^k}.Z\{u_n\},$$

where $\{u_{n-k}\}$ denotes the sequence whose first k terms, corresponding to
$n = 0,\ 1,\ 2,\ \ldots.,k-1$, are taken as zero and whose subsequent terms, corresponding to
$n = k,\ k+1,\ k+2,\ \ldots.$ are the terms $u_0,\ u_1,\ u_2,\ \ldots.$ of the original sequence.

ILLUSTRATION

Given that $\{u_n\} \equiv \{4^n\}$, we may say that

$$Z\{u_{n-2}\} \equiv \frac{1}{z^2}.Z\{u_n\} \equiv \frac{1}{z^2}.\frac{z}{z-4} \equiv \frac{1}{z(z-4)}.$$

Note:
$\{u_{n-2}\}$ has terms $0,\ 0,\ 1,\ 4,\ 4^2,\ 4^3,\ \ldots$ and, by applying the definition of a Z-Transform directly, we would obtain

$$Z\{u_{n-2}\} = \frac{1}{z^2} + \frac{4}{z^3} + \frac{4^2}{z^4} + \frac{4^3}{z^5}\ \ldots.$$

$$\text{Hence,} \quad Z\{u_n\} \equiv \frac{1}{z^2}.\frac{1}{1-\frac{4}{z}} \equiv \frac{1}{z(z-4)},$$

by properties of infinite geometric series

(c) The Second Shifting Theorem

$$Z\{u_{n+1}\} \equiv z.Z\{u_n\} - z.u_0$$

Proof:

The left-hand side is equivalent to

$$\sum_{r=0}^{\infty} u_{r+1}z^{-r} \equiv u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \frac{u_4}{z^4} + \ \ldots .$$

That is,

$$z.\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \ \ldots .\right] - z.u_0$$

This is equivalent to the right-hand side

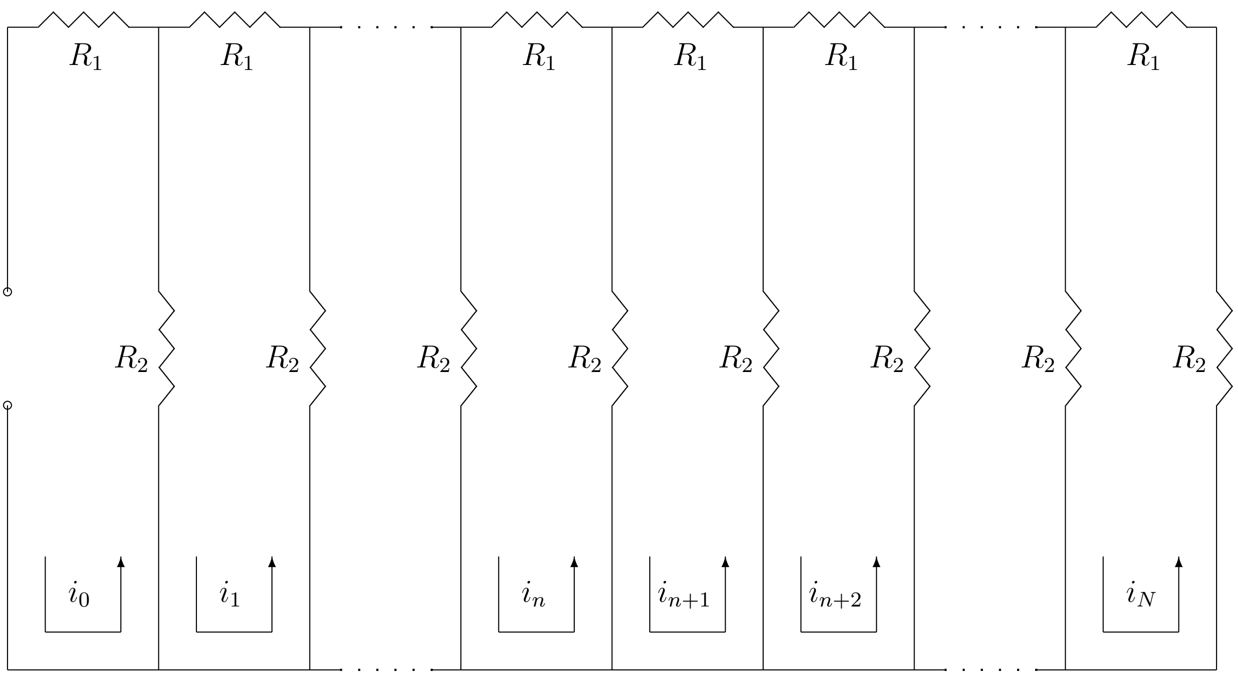
Note:

$$Z\{u_{n+2}\} \equiv z.Z\{u_{n+1}\} - z.u_1 \equiv z^2.Z\{u_n\} - z^2.u_0 - z.u_1$$

Z-TRANSFORMS
(AN ENGINEERING INTRODUCTION)

The mathematics of certain engineering problems leads to what are know as **"difference equations"**

For example, consider the following electrical **"ladder network"**.



It may be shown that

$$R_1 i_{n+1} + R_2(i_{n+1} - i_n) + R_2(i_{n+1} - i_{n+2}) = 0$$

where $0 \leq n \leq N - 2$.

The question which arises is "how can we determine a formula for i_n in terms of n" ?