"JUST THE MATHS"

SLIDES NUMBER

16.7

LAPLACE TRANSFORMS 7
(An appendix)

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One view of how Laplace Transforms might have arisen

UNIT 16.7 - LAPLACE TRANSFORMS 7 (AN APPENDIX)

ONE VIEW OF HOW LAPLACE TRANSFORMS MIGHT HAVE ARISEN.

(i) The problem is to solve a second order linear differential equation with constant coefficients,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

(ii) We assume that the equivalent first order differential equation,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

has already been studied.

We examine the following example:

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}x}{\mathrm{d}t} + 3x = e^{2t},$$

given that $x = 0$ when $t = 0$.

Solution

The **"integrating factor method"** uses the coefficient of x to find a function of t which multiplies both sides of the given differential equation to convert it to an **"exact"** differential equation

The integrating factor in the current example is e^{3t}, since the coefficient of x is 3.

We obtain,

$$e^{3t}\left[\frac{\mathrm{d}x}{\mathrm{d}t}+3x\right]=e^{5t}.$$

This is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right]=e^{5t}.$$

Integrating both sides with respect to t,

$$xe^{3t}=\frac{e^{5t}}{5}+C$$

or

$$x=\frac{e^{2t}}{5}+Ce^{-3t}.$$

Putting $x = 0$ and $t = 0$, we have

$$0 = \frac{1}{5} + C.$$

Hence, $C = -\frac{1}{5}$, and the complete solution becomes

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}.$$

(iii) We shall now examine a different way of setting out the above working in which the boundary condition is substituted earlier.

We multiply both sides of the differential equation by e^{3t} as before, then integrate both sides of the new "exact" equation from 0 to t.

$$\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\left[xe^{3t}\right]\,\mathrm{d}t = \int_0^t e^{5t}\,\mathrm{d}t.$$

That is,

$$\left[xe^{3t}\right]_0^t = \left[\frac{e^{5t}}{5}\right]_0^t,$$

giving

$$xe^{3t} - 0 = \frac{e^{5t}}{5} - \frac{1}{5} \quad \text{since} \quad x = 0 \quad \text{when} \quad t = 0.$$

In other words,

$$x = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5},$$

as before

(iv) We consider, next, whether an example of a second order linear differential equation could be solved by a similar method.

EXAMPLE

Solve the differential equation,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x = e^{9t},$$

given that $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$ when $t = 0$.

Solution

We assume that an integrating factor for this equation is e^{st}, where s, at present, is unknown, but is assumed to be positive

Hence, we multiply throughout by e^{st} and integrate from 0 to t.

$$\int_0^t e^{st}\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - 10\frac{\mathrm{d}x}{\mathrm{d}t} + 21x\right] \mathrm{d}t = \int_0^t e^{(s+9)t} \mathrm{d}t$$

$$= \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

Using **"integration by parts"**, and the boundary condition,

$$\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t = e^{st}x - s\int_0^t e^{st}x \mathrm{d}t.$$

$$\int_0^t e^{st}\frac{\mathrm{d}^2x}{\mathrm{d}t^2} \mathrm{d}t = e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - s\int_0^t e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$$

$$= e^{st}\frac{\mathrm{d}x}{\mathrm{d}t} - se^{st}x + s^2\int_0^t e^{st}x \mathrm{d}t.$$

Substituting these results into the differential equation, we may collect together terms which involve

$$\int_0^t e^{st}x \mathrm{d}t \quad \text{and} \quad e^{st}$$

as follows:

$$(s^2+10s+21)\int_0^t e^{st}x \mathrm{d}t + e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right] = \left[\frac{e^{(s+9)t}}{s+9}\right]_0^t.$$

(v) OBSERVATIONS

(a) If we had used e^{-st} instead of e^{st}, the quadratic expression in s, above, would have had the same coefficients as the original differential equation.

That is,

$$(s^2 - 10s + 21).$$

(b) Using e^{-st} with $s > 0$, suppose we had integrated from 0 to ∞ instead of 0 to t.

The term,

$$e^{st}\left[\frac{\mathrm{d}x}{\mathrm{d}t} - (s+10)x\right],$$

would have been absent, since $e^{-\infty} = 0$.

(vi) Having made our observations, we start again, multiplying both sides of the differential equation by e^{-st} and integrating from 0 to ∞.

We obtain

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x\ \mathrm{d}t = \left[\frac{e^{(-s+9)t}}{-s+9}\right]_0^\infty.$$

$$(s^2 - 10s + 21)\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{-1}{-s+9} = \frac{1}{s-9}.$$

Note:
This works only if $s > 9$, but we can easily assume that it is so.

Using $s^2 - 10s + 21 \equiv (s-3)(s-7)$,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{1}{(s-9)(s-3)(s-7)}.$$

Using partial fractions,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t = \frac{1}{12}.\frac{1}{s-9} + \frac{1}{24}.\frac{1}{s-3} - \frac{1}{8}.\frac{1}{s-7}.$$

(vii) Finally, it can be shown, by an independent method of solution, that

$$x = \frac{e^{9t}}{12} + \frac{e^{3t}}{24} - \frac{e^{7t}}{8}.$$

We may conclude that the solution of the differential equation is closely linked to the integral,

$$\int_0^\infty e^{-st}x \, \mathrm{d}t,$$

which is called the **"Laplace Transform"** of $x(t)$.