"JUST THE MATHS"

SLIDES NUMBER

16.6

LAPLACE TRANSFORMS 6
(The Dirac unit impulse function)

by

A.J.Hobson

UNIT 16.6 - LAPLACE TRANSFORMS 6

THE DIRAC UNIT IMPULSE FUNCTION

16.6.1 THE DEFINITION OF THE DIRAC UNIT IMPULSE FUNCTION

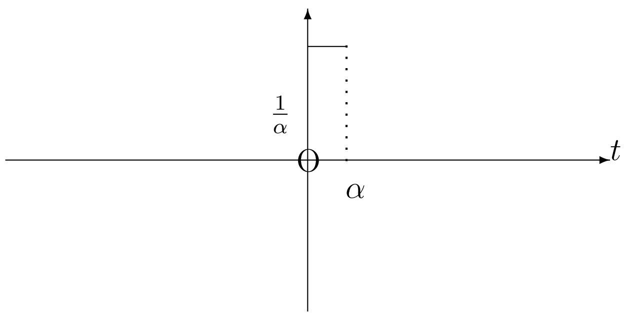
DEFINITION 1

A pulse of large magnitude, short duration and finite strength is called an **"impulse"**.

A **"unit impulse"** is an impulse of strength 1.

ILLUSTRATION

Consider a pulse, of duration α, between $t = 0$ and $t = \alpha$, having magnitude $\frac{1}{\alpha}$. The strength of the pulse is then 1.



Using Heaviside step functions, this pulse is given by

$$\frac{H(t) - H(t - \alpha)}{\alpha}.$$

Allowing α to tend to zero, we obtain a unit impulse located at $t = 0$.

DEFINITION 2

The **"Dirac unit impulse function"** , $\delta(t)$, is defined to be an impulse of unit strength, located at $t = 0$.

It is given by

$$\delta(t) = \lim_{\alpha \to 0} \frac{H(t) - H(t - \alpha)}{\alpha}.$$

Notes:

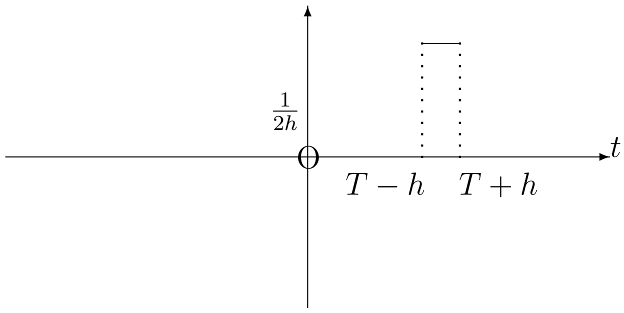
(i) An impulse of unit strength located at $t = T$ is represented by $\delta(t - T)$.

(ii) An alternative definition of the function $\delta(t - T)$ is as follows:

$$\delta(t - T) = \begin{cases} 0 & \text{for } t \neq T; \\ \infty & \text{for } t = T \end{cases}$$

and

$$\lim_{h \to 0} \int_{T-h}^{T+h} \delta(t - T) \ \mathrm{d}t = 1.$$



THEOREM

$$\int_a^b f(t)\delta(t-T)\ \mathrm{d}t = f(T) \quad \text{if} \quad a < T < b.$$

Proof:

Since $\delta(t-T)$ is equal to zero everywhere except at $t = T$, the left-hand side of the above formula reduces to

$$\lim_{h\to 0} \int_{T-h}^{T+h} f(t)\delta(t-T)\ \mathrm{d}t$$

But in the small interval from $T-h$ to $T+h$, $f(t)$ is approximately constant and equal to $f(T)$.

Hence, the left-hand side may be written

$$f(T)\left[\lim_{h\to 0} \int_{T-h}^{T+h} \delta(T-T)\ \mathrm{d}t\right].$$

This reduces to $f(T)$, using note (ii) earlier

16.6.2 THE LAPLACE TRANSFORM OF THE DIRAC UNIT IMPULSE FUNCTION

RESULT

$$L[\delta(t-T)] = e^{-sT};$$

and, in particular, $L[\delta(t)] = 1$.

Proof:

From the definition of a Laplace Transform,

$$L[\delta(t-T) = \int_0^\infty e^{-st}\delta(t-T)\ \mathrm{d}t.$$

But, from the Theorem just discussed,
with $f(t) = e^{-st}$,

$$L[\delta(t-T)] = e^{-sT}.$$

EXAMPLES

1. Solve the differential equation

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = \delta(t),$$

given that $x = 0$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$3sX(s) + 4X(s) = 1.$$

That is,

$$X(s) = \frac{1}{3s+4} = \frac{1}{3}.\frac{1}{s+\frac{4}{3}}.$$

Hence,

$$x(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

2. Show that, for any function, $f(t)$,

$$\int_0^\infty f(t)\delta'(t-a)\ \mathrm{d}t = -f'(a).$$

Solution

Using Integration by Parts, the left-hand side of the formula may be written

$$[f(t)\delta(t-a)]_0^\infty - \int_0^\infty f'(t)\delta(t-a)\ \mathrm{d}t.$$

The first term of this reduces to zero, since $\delta(t-a)$ is equal to zero except when $t = a$.

The required result follows from the Theorem discussed earlier, with $T = a$.

16.6.3 TRANSFER FUNCTIONS

In scientific applications, the solution of

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

is sometimes called the **"response of a system to the function $f(t)$"**.

The term **"system"** may, for example, refer to an oscillatory electrical circuit or a mechanical vibration.

We may refer to $f(t)$ as the **"input"** and $x(t)$ as the **"output"** of a system.

In the work which follows, we shall consider the special case in which $x = 0$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 0$, when $t = 0$

That is, we shall assume zero initial conditions.

Impulse response and transfer function

Consider, first, the differential equation

$$a\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + b\frac{\mathrm{d}u}{\mathrm{d}t} + cu = \delta(t).$$

We refer to the function, $u(t)$, as the **"impulse response function"** of the original system.

The Laplace Transform of its differential equation is given by

$$(as^2 + bs + c)U(s) = 1.$$

Hence,

$$U(s) = \frac{1}{as^2 + bs + c}.$$

This is called the **"transfer function"** of the original system.

EXAMPLE

Determine the transfer function and impulse response function for the differential equation,

$$3\frac{\mathrm{d}x}{\mathrm{d}t} + 4x = f(t),$$

assuming zero initial conditions.

Solution

To find $U(s)$ and $u(t)$, we have

$$3\frac{\mathrm{d}u}{\mathrm{d}t} + 4u = \delta t,$$

so that

$$(3s + 4)U(s) = 1.$$

Hence, the transfer function is

$$U(s) = \frac{1}{3s + 4} = \frac{1}{3}.\frac{1}{s + \frac{4}{3}}.$$

Taking the inverse Laplace Transform of $U(s)$ gives the impulse response function,

$$u(t) = \frac{1}{3}e^{-\frac{4t}{3}}.$$

System Response for any Input

Assuming zero initial conditions, the Laplace Transform of the differential equation,

$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + bx + cx = f(t),$$

is given by

$$(as^2 + bs + c)X(s) = F(s).$$

Thus,

$$X(s) = \frac{F(s)}{as^2 + bs + c} = F(s).U(s).$$

In order to find the response of the system to the function $f(t)$, we need the inverse Laplace Transform of $F(s).U(s)$.

This may possibly be found using partial fractions; but it may, if necessary, be found by using the Convolution Theorem (Unit 16.1)

The Convolution Theorem shows, in this case, that

$$L\left[\int_0^t f(T).u(t-T)\ \mathrm{d}T\right] = F(s).U(s).$$

In other words,

$$L^{-1}[F(s).U(s)] = \int_0^t f(T).u(t-T)\ \mathrm{d}T.$$

EXAMPLE

The impulse response of a system is known to be

$$u(t) = \frac{10e^{-t}}{3}.$$

Determine the response, $x(t)$, of the system to an input of

$$f(t) \equiv \sin 3t.$$

Solution

First, we note that

$$U(s) = \frac{10}{3(s+1)} \ \text{ and } \ F(s) = \frac{3}{s^2+9}.$$

Hence,

$$X(s) = \frac{10}{(s+1)(s^2+9)} = \frac{1}{s+1} + \frac{-s+1}{s^2+9}.$$

Thus,

$$x(t) = e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0.$$

Alternatively, using the Convolution Theorem,

$$x(t) = \int_0^t \sin 3T.\frac{10e^{-(t-T)}}{3}\, \mathrm{d}T.$$

The integration here may be made simpler if we replace $\sin 3T$ by e^{j3T} and use the imaginary part, only, of the result.

Hence,

$$x(t) = I_m\left(\int_0^t \frac{10}{3}e^{-t}.e^{(1+j3)T}\, \mathrm{d}T\right)$$

$$= I_m\left(\frac{10}{3}\left[e^{-t}\frac{e^{(1+j3)T}}{1+j3}\right]_0^t\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{e^{-t}.e^{(1+j3)t} - e^{-t}}{1+j3}\right]\right)$$

$$= I_m\left(\frac{10}{3}\left[\frac{[(\cos 3t - e^{-t}) + j\sin 3t](1-j3)}{10}\right]\right)$$

$$= \frac{10}{3}\left[\frac{\sin 3t - 3\cos 3t + 3e^{-t}}{10}\right]$$

$$= e^{-t} - \cos 3t + \frac{1}{3}\sin 3t \quad t > 0,$$

as before.

Note:
In this example, the method using partial fractions is simpler.

16.6.4 STEADY-STATE RESPONSE TO A SINGLE FREQUENCY INPUT

Consider the differential equation

$$a\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t).$$

Suppose that the quadratic denominator of the transfer function, $U(s)$, has negative real roots.

That is, the impulse response, $u(t)$, contains negative powers of e and, hence, tends to zero as t tends to infinity.

Suppose also that $f(t)$ is either $\cos \omega t$ or $\sin \omega t$.

These may be regarded, respectively, as the real and imaginary parts of the function $e^{j\omega t}$.

It may be shown that the response, $x(t)$, will consist of a **"transient"** part which tends to zero as t tends to infinity together with a non-transient part forming the **"steady-state response"**.

We illustrate with an example:

EXAMPLE

Consider the equation

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 3\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = e^{j7t},$$

where $x = 2$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = 1$ when $t = 0$.

Solution

Taking the Laplace Transform of the differential equation,

$$s(sX(s) - 2) - 1 + 3(sX(s) - 2) + 2X(s) = \frac{1}{s - j7}.$$

That is,

$$(s^2 + 3s + 2)X(s) = 2s + 7 + \frac{1}{s - j7},$$

giving

$$X(s) = \frac{2s + 7}{s^2 + 3s + 2} + \frac{1}{(s - j7)(s^2 + 3s + 2)}$$

$$= \frac{2s + 7}{(s + 2)(s + 1)} + \frac{1}{(s - j7)(s + 2)(s + 1)}.$$

Using partial fractions,

$$X(s) = \frac{5}{s + 1} - \frac{3}{s + 2}$$

$$+\frac{1}{(-1 - j7)(s + 1)} + \frac{1}{(2 + j7)(s + 2)} + \frac{U(j7)}{(s - j7)},$$

where

$$U(s) \equiv \frac{1}{s^2 + 3s + 2}$$

is the transfer function.

Taking inverse Laplace Transforms,

$$x(t) = 5e^{-t} - 3e^{-2t}$$

$$+\frac{1}{-1-j7}e^{-t} + \frac{1}{2+j7}e^{-2t} + U(j7)e^{j7t}.$$

The first four terms on the right-hand side tend to zero as t tends to infinity.

The final term represents the steady state response.

We need its real part if $f(t) \equiv \cos 7t$ and its imaginary part if $f(t) \equiv \sin 7t$.

In this example

$$U(j7) = \frac{1}{-47 + j21} = \frac{-47}{2650} - j\frac{21}{2650}.$$

Summary

The above example illustrates the result that the steady-state response, $s(t)$, of a system to an input of $e^{j\omega t}$ is given by

$$s(t) = U(j\omega)e^{j\omega t}.$$