"JUST THE MATHS"

SLIDES NUMBER

16.5

LAPLACE TRANSFORMS 5
(The Heaviside step function)

by

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UNIT 16.5 - LAPLACE TRANSFORMS 5

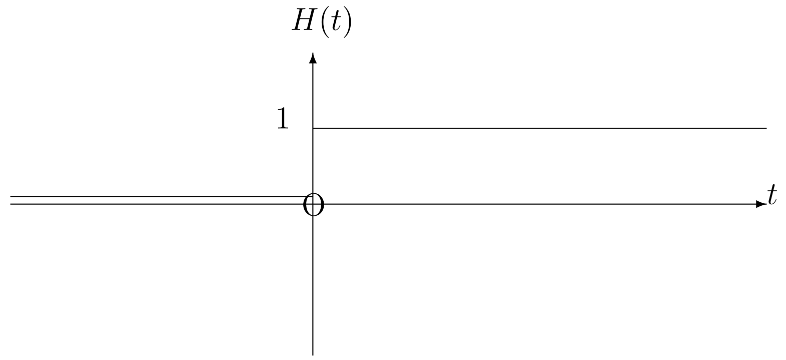
THE HEAVISIDE STEP FUNCTION

16.5.1 THE DEFINITION OF THE HEAVISIDE STEP FUNCTION

The **"Heaviside step function"**, $H(t)$, is defined by the statements,

$$H(t) = \begin{cases} 0 & \text{for } t < 0; \\ 1 & \text{for } t > 0. \end{cases}$$

Note: $H(t)$ is undefined when $t = 0$.

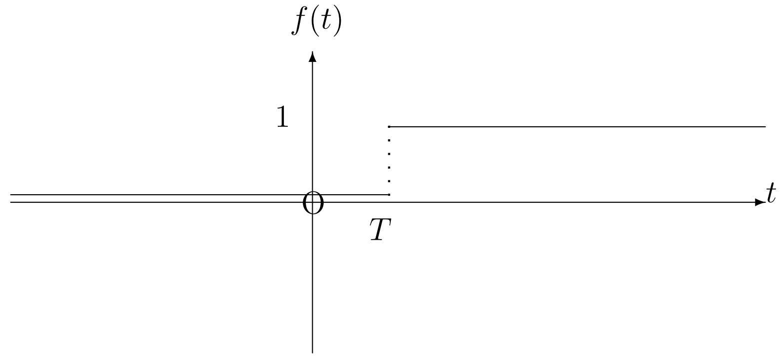


EXAMPLE

Express, in terms of $H(t)$, the function, $f(t)$, given by the statements

$$f(t) = \begin{cases} 0 & \text{for } t < T; \\ 1 & \text{for } t > T. \end{cases}$$

Solution



$f(t)$ is the same type of function as $H(t)$, but we have effectively moved the origin to the point $(T, 0)$.

Hence,

$$f(t) \equiv H(t - T).$$

Note:
The function, $H(t - T)$, is of importance in constructing what are known as **"pulse functions"**.

16.5.2 THE LAPLACE TRANSFORM OF H(t - T)

From the definition of a Laplace Transform,

$$L[H(t-T)] = \int_0^\infty e^{-st} H(t-T)\ \mathrm{d}t$$

$$= \int_0^T e^{-st}.0\ \mathrm{d}t + \int_T^\infty e^{-st}.1\ \mathrm{d}t$$

$$= \left[\frac{e^{-st}}{-s}\right]_T^\infty = \frac{e^{-sT}}{s}.$$

Note:
In the special case when $T = 0$,

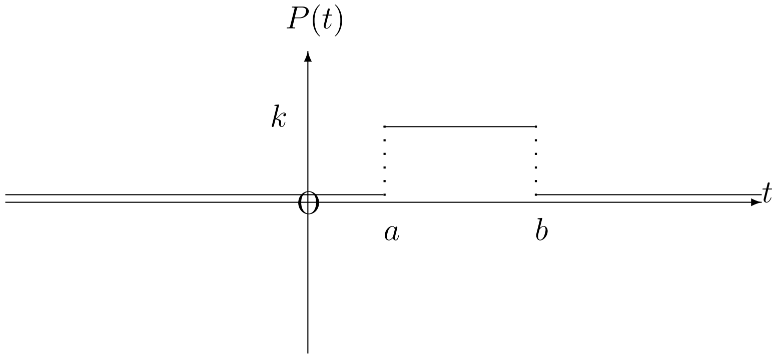
$$L[H(t)] = \frac{1}{s}.$$

This can be expected, since $H(t)$ and 1 are identical over the range of integration.

16.5.3 PULSE FUNCTIONS

If $a < b$, a **"rectangular pulse"**, $P(t)$, of duration $b - a$ and magnitude, k, is defined by the statements,

$$P(t) = \begin{cases} k & \text{for } a < t < b; \\ 0 & \text{for } t < a \text{ or } t > b. \end{cases}$$



In terms of Heaviside functions,

$$P(t) \equiv k[H(t-a) - H(t-b)].$$

Proof:

(i) If $t < a$, then $H(t-a) = 0$ and $H(t-b) = 0$.

Hence, the above right-hand side = 0.

(ii) If $t > b$, then $H(t-a) = 1$ and $H(t-b) = 1$.

Hence, the above right-hand side = 0.

(iii) If $a < t < b$, then $H(t-a) = 1$ and $H(t-b) = 0$.

Hence, the above right-hand side $= k$.

EXAMPLE

Determine the Laplace Transform of a pulse, $P(t)$, of duration $b - a$, having magnitude, k.

Solution

$$L[P(t)] = k\left[\frac{e^{-sa}}{s} - \frac{e^{-sb}}{s}\right]$$

$$= k.\frac{e^{-sa} - e^{-sb}}{s}.$$

Notes:

(i) The **"strength"** of the pulse described above is defined as the area of the rectangle with base, $b - a$, and height, k.

That is,

$$\text{strength} = k(b - a).$$

(ii) The expression,

$$[H(t-a) - H(t-b)]f(t),$$

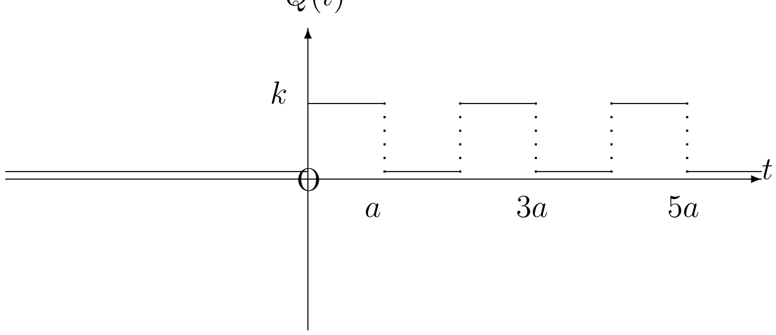
can be considered to **"switch on"** the function, $f(t)$, between $t = a$ and $t = b$ but **"switch off"** the function, $f(t)$, when $t < a$ or $t > b$.

(iii) The expression,

$$H(t-a)f(t),$$

can be considered to **"switch on"** the function, $f(t)$, when $t > a$ but **"switch off"** the function, $f(t)$, when $t < a$.

For example, consider the train of rectangular pulses, $Q(t)$, in the following diagram:



The graph can be represented by the function

$$k\{[H(t) - H(t-a)] + [H(t-2a) - H(t-3a)]$$

$$+[H(t-4a) - H(t-5a)] + ...\}$$

16.5.4 THE SECOND SHIFTING THEOREM

THEOREM

$$L[H(t-T)f(t-T)] = e^{-sT}L[f(t)].$$

Proof:

Left-hand side =

$$\int_0^\infty e^{-st}H(t-T)f(t-T)\ \mathrm{d}t$$

$$= \int_0^T 0\ \mathrm{d}t + \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t$$

$$= \int_T^\infty e^{-st}f(t-T)\ \mathrm{d}t.$$

Making the substitution, $u = t - T$, gives

$$\int_0^\infty e^{-s(u+T)}f(u)\ \mathrm{d}u$$

$$= e^{-sT}\int_0^\infty e^{-su}f(u)\ \mathrm{d}u = e^{-sT}L[f(t)].$$

EXAMPLES

1. Express, in terms of Heaviside functions, the function

$$f(t) = \begin{cases} (t-1)^2 & \text{for } t > 1; \\ 0 & \text{for } 0 < t < 1 \end{cases}$$

and, hence, determine its Laplace Transform.

Solution

For values of $t > 0$, we can write

$$f(t) = (t-1)^2 H(t-1).$$

Using $T = 1$ in the second shifting theorem,

$$L[f(t)] = e^{-s} L[t^2] = e^{-s} . \frac{2}{s^3}.$$

2. Determine the inverse Laplace Transform of the expression,

$$\frac{e^{-7s}}{s^2 + 4s + 5}.$$

Solution

First, we find the inverse Laplace Transform of the expression

$$\frac{1}{s^2 + 4s + 5} \equiv \frac{1}{(s + 2)^2 + 1}.$$

From the first shifting theorem, this will be the function

$$e^{-2t} \sin t, \quad t > 0.$$

From the second shifting theorem, the required function will be

$$H(t - 7)e^{-2(t-7)} \sin(t - 7), \quad t > 0.$$