# “JUST THE MATHS”

# SLIDES NUMBER

# 16.4

# LAPLACE TRANSFORMS 4
# (Simultaneous differential equations)

## by



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# UNIT 16.4 - LAPLACE TRANSFORMS 4

## 16.4.1 AN EXAMPLE OF SOLVING SIMULTANEOUS LINEAR DIFFERENTIAL EQUATIONS

We consider, here, a pair of differential equations of the form

$$
\begin{aligned}
a_1\frac{\mathrm{d}x}{\mathrm{d}t} + b_1\frac{\mathrm{d}y}{\mathrm{d}t} + c_1x + d_1y &= f_1(t),\\
a_2\frac{\mathrm{d}x}{\mathrm{d}t} + b_2\frac{\mathrm{d}y}{\mathrm{d}t} + c_2x + d_2y &= f_2(t).
\end{aligned}
$$

To solve these equations simultaneously, we take the Laplace Transform of each equation.

This leads to two simultaneous algebraic equations from which we may determine $X(s)$ and $Y(s)$, the Laplace Transforms of $x(t)$ and $y(t)$ respectively.

## EXAMPLE

Solve, simultaneously, the differential equations

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} + 2x &= e^t, \\ \frac{\mathrm{d}x}{\mathrm{d}t} - 2y &= 1 + t,\end{aligned}$$

given that $x(0) = 1$ and $y(0) = 2$.

## Solution

Taking the Laplace Transforms of the differential equations

$$\begin{aligned}sY(s) - 2 + 2X(s) &= \frac{1}{s-1}, \\ sX(s) - 1 - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2}.\end{aligned}$$

That is,

$$\begin{aligned}2X(s) + sY(s) &= \frac{1}{s-1} + 2, \qquad (1) \\ sX(s) - 2Y(s) &= \frac{1}{s} + \frac{1}{s^2} + 1. \qquad (2)\end{aligned}$$

Using (1) $\times$ 2 + (2) $\times$ $s$, we obtain

$$(4+s^2)X(s) = \frac{2}{s-1} + 4 + 1 + \frac{1}{s} + s.$$

Hence,

$$X(s) = \frac{2}{(s-1)(s^2+4)} + \frac{5}{s^2+4} + \frac{1}{s(s^2+4)} + \frac{s}{s^2+4}.$$

Using partial fractions, it may be shown that

$$X(s) = \frac{2}{5}.\frac{1}{s-1} + \frac{7}{20}.\frac{s}{s^2+4} + \frac{23}{5}.\frac{1}{s^2+4} + \frac{1}{4}.\frac{1}{s}.$$

Thus,

$$x(t) = \frac{2}{5}e^t + \frac{1}{4} + \frac{7}{20}\cos 2t + \frac{23}{10}\sin 2t \quad t > 0.$$

We could now start again by eliminating $x$ from equations (1) and (2) in order to calculate $y$.

However, in this example,

$$2y = \frac{\mathrm{d}x}{\mathrm{d}t} - 1 - t; \quad \text{and so.}$$

$$y(t) = \frac{1}{5}e^t - \frac{1}{2} - \frac{7}{20}\sin 2t + \frac{23}{10}\cos 2t - \frac{t}{2} \quad t > 0.$$