"JUST THE MATHS"

SLIDES NUMBER

16.2

LAPLACE TRANSFORMS 2
(Inverse Laplace Transforms)

by

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UNIT 16.2 - LAPLACE TRANSFORMS 2

16.2.1 THE DEFINITION OF AN INVERSE LAPLACE TRANSFORM

A function of t, whose Laplace Transform is $F(s)$, is called the **"Inverse Laplace Transform"** of $F(s)$ and may be denoted by the symbol

$$L^{-1}[F(s)].$$

Notes:

(i) Two functions which coincide for $t > 0$ will have the same Laplace Transform, so we can determine $L^{-1}[F(s)]$ only for **positive** values of t.

(ii) Inverse Laplace Transforms are **linear**.

Proof:

$L^{-1}\left[A.F(s) + B.G(s)\right]$ is a function of t whose Laplace Transform is $A.F(s) + B.G(s)$.

By the linearity of Laplace Transforms, such a function is

$$A.L^{-1}[F(s)] + B.L^{-1}[G(s)].$$

16.2.2 METHODS OF DETERMINING AN INVERSE LAPLACE TRANSFORM

We consider problems where the Laplace Transforms are **"rational functions of** s**"**.

Partial fractions will be used.

EXAMPLES

1. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{s^3} + \frac{4}{s-2}.$$

Solution

$$f(t) = \frac{3}{2}t^2 + 4e^{2t} \quad t > 0$$

2. Determine the Inverse Laplace Transform of

$$F(s) = \frac{2s+3}{s^2+3s} = \frac{2s+3}{s(s+3)}.$$

Solution

Using partial fractions,

$$\frac{2s+3}{s(s+3)} \equiv \frac{A}{s} + \frac{B}{s+3},$$

giving

$$2s + 3 \equiv A(s+3) + Bs.$$

Note:
Although the s of a Laplace Transform is an arbitrary **positive** number, we may temporarily ignore that in order to complete the partial fractions.

Putting $s = 0$ and $s = -3$ gives

$$3 = 3A \text{ and } -3 = -3B.$$

Thus,

$$A = 1 \text{ and } B = 1.$$

Hence,

$$F(s) = \frac{1}{s} + \frac{1}{s+3}.$$

Finally,

$$f(t) = 1 + e^{-3t} \quad t > 0.$$

3. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{s^2 + 9}.$$

Solution

$$f(t) = \frac{1}{3}\sin 3t \quad t > 0.$$

4. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+2}{s^2 + 5}.$$

Solution

$$f(t) = \cos t\sqrt{5} + \frac{2}{\sqrt{5}} \sin t\sqrt{5} \quad t > 0.$$

5. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3s^2 + 2s + 4}{(s+1)(s^2+4)}.$$

Solution

Using partial fractions,

$$\frac{3s^2 + 2s + 4}{(s+1)(s^2+4)} \equiv \frac{A}{s+1} + \frac{Bs + C}{s^2+4}.$$

That is,

$$3s^2 + 2s + 4 \equiv A(s^2+4) + (Bs + C)(s+1).$$

Substituting $s = -1$,

$$5 = 5A \text{ implying that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$3 = A + B \text{ so that } B = 2.$$

Equating constant terms on both sides,

$$4 = 4A + C \text{ so that } C = 0.$$

We conclude that

$$F(s) = \frac{1}{s+1} + \frac{2s}{s^2+4}.$$

Hence,

$$f(t) = e^{-t} + 2\cos 2t \quad t > 0.$$

6. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s+2)^5}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{n!}{s^{n+1}}$, we obtain

$$f(t) = \frac{1}{24} t^4 e^{-2t} \quad t > 0.$$

7. Determine the Inverse Laplace Transform of

$$F(s) = \frac{3}{(s-7)^2 + 9}.$$

Solution

Using the First Shifting Theorem and the Inverse Laplace Transform of $\frac{a}{s^2+a^2}$, we obtain

$$f(t) = e^{7t} \sin 3t \quad t > 0.$$

8. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s}{s^2 + 4s + 13}.$$

Solution

The denominator will not factorise conveniently, so we **complete the square**.

This gives

$$F(s) = \frac{s}{(s+2)^2 + 9}.$$

To use the First Shifting Theorem, we must include $s+2$ in the numerator.

Thus, we write

$$F(s) = \frac{(s+2)-2}{(s+2)^2+9} = \frac{s+2}{(s+2)^2+3^2} - \frac{2}{3}.\frac{3}{(s+2)^2+3^2}.$$

Hence, for $t > 0$,

$$f(t) = e^{-2t}\cos 3t - \frac{2}{3}e^{-2t}\sin 3t = \frac{1}{3}e^{-2t}\left[3\cos 3t - 2\sin 3t\right].$$

9. Determine the Inverse Laplace Transform of

$$F(s) = \frac{8(s+1)}{s(s^2+4s+8)}.$$

Solution

Using partial fractions,

$$\frac{8(s+1)}{s(s^2+4s+8)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+4s+8}.$$

Mutiplying up, we obtain

$$8(s+1) \equiv A(s^2+4s+8) + (Bs+C)s.$$

Substituting $s = 0$ gives

$$8 = 8A \text{ so that } A = 1.$$

Equating coefficients of s^2 on both sides,

$$0 = A + B \text{ which gives } B = -1.$$

Equating coefficients of s on both sides,

$$8 = 4A + C \text{ which gives } C = 4.$$

Thus,

$$F(s) = \frac{1}{s} + \frac{-s+4}{s^2+4s+8}.$$

The quadratic denominator will not factorise conveniently, so we complete the square.

This gives

$$F(s) = \frac{1}{s} + \frac{-s+4}{(s+2)^2+4},$$

On rearrangement,

$$F(s) = \frac{1}{s} - \frac{s+2}{(s+2)^2+2^2} + \frac{6}{(s+2)^2+2^2}.$$

From the First Shifting Theorem,

$$f(t) = 1 - e^{-2t}\cos 2t + 3e^{-2t}\sin 2t \quad t > 0.$$

10. Determine the Inverse Laplace Transform of

$$F(s) = \frac{s+10}{s^2-4s-12}.$$

Solution

This time, the denominator **will** factorise, into $(s+2)(s-6)$.

Partial fractions give

$$\frac{s+10}{(s+2)(s-6)} \equiv \frac{A}{s+2} + \frac{B}{s-6}.$$

Hence,

$$s+10 \equiv \mathrm{A}(s-6) + \mathrm{B}(s+2).$$

Putting $s=-2$,

$$8 = -8A \text{ giving } A = -1.$$

Putting $s=6$,

$$16 = 8B \text{ giving } B = 2.$$

We conclude that

$$F(s) = \frac{-1}{s+2} + \frac{2}{s-6}.$$

Finally,

$$f(t) = -e^{-2t} + 2e^{6t} \quad t > 0.$$

Note:

If we did not factorise the denominator,

$$F(s) = \frac{(s-2)+12}{(s-2)^2-4^2} = \frac{s-2}{(s-2)^2-4^2} + 3.\frac{4}{(s-2)^2+4^2}.$$

Hence,

$$f(t) = e^{2t}[\cosh 4t + 3\sinh 4t] \quad t > 0.$$

11. Determine the Inverse Laplace Transform of

$$F(s) = \frac{1}{(s-1)(s+2)}.$$

Solution

The Inverse Laplace Transform could certainly be obtained by using partial fractions.

But also, it could be obtained from the Convolution Theorem.

Writing

$$F(s) = \frac{1}{(s-1)} \cdot \frac{1}{(s+2)},$$

we obtain

$$f(t) = \int_0^t e^T . e^{-2(t-T)} \, \mathrm{d}T = \int_0^t e^{(3T-2t)} \, \mathrm{d}T = \left[\frac{e^{3T-2t}}{3} \right]_0^t .$$

That is,

$$f(t) = \frac{e^t}{3} - \frac{e^{-2t}}{3} \quad t > 0.$$