"JUST THE MATHS"

SLIDES NUMBER

16.10

Z-TRANSFORMS 3
(Solution of linear difference equations)

by

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UNIT 16.10 - Z TRANSFORMS 3

THE SOLUTION OF LINEAR DIFFERENCE EQUATIONS

Linear Difference Equations may be solved by constructing the Z-Transform of both sides of the equation.

16.10.1 FIRST ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+1} - 2u_n = (3)^{-n},$$

given that $u_0 = 2/5$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.\frac{2}{5}.$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - \frac{2}{5}.z - 2Z\{u_n\} = \frac{z}{z - \frac{1}{3}}.$$

On rearrangement,

$$Z\{u_n\} = \frac{2}{5}.\frac{z}{z-2} + \frac{z}{\left(z-\frac{1}{3}\right)(z-2)}$$

$$\equiv \frac{2}{5}.\frac{z}{z-2} + z.\left[\frac{-\frac{3}{5}}{z-\frac{1}{3}} + \frac{\frac{3}{5}}{z-2}\right]$$

$$\equiv \frac{z}{z-2} - \frac{3}{5}.\frac{z}{z-\frac{1}{3}}.$$

Taking the inverse Z-Transform of this function of z,

$$\{u_n\} \equiv \left\{(2)^n - \frac{3}{5}(3)^{-n}\right\}.$$

2. Solve the linear difference equation,

$$u_{n+1} + u_n = f(n),$$

given that

$$f(n) \equiv \begin{cases} 1 & \text{when } n = 0; \\ 0 & \text{when } n > 0. \end{cases}$$

and $u_0 = 5$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - z.5$$

Taking the Z-Transform of the difference equation,

$$z.Z\{u_n\} - 5z + Z\{u_n\} = 1.$$

On rearrangement,

$$Z\{u_n\} = \frac{1}{z+1} + \frac{5z}{z+1}.$$

Hence,

$$\{u_n\} = \begin{cases} 5 & \text{when } n = 0; \\ (-1)^{n-1} + 5(-1)^n \equiv 4(-1)^n & \text{when } n > 0. \end{cases}$$

16.10.2 SECOND ORDER LINEAR DIFFERENCE EQUATIONS

EXAMPLES

1. Solve the linear difference equation,

$$u_{n+2} = u_{n+1} + u_n,$$

given that $u_0 = 0$ and $u_1 = 1$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n - z.0 \equiv z.Z\{u_n\}.$$

and

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z.1 \equiv z^2 Z\{u_n\} - z.$$

Taking the Z-Transform of the difference equation,

$$z^2.Z\{u_n\} - z = z.Z\{u_n\} + Z\{u_n\}.$$

On rearrangement,

$$Z\{u_n\} = \frac{z}{z^2 - z - 1}.$$

This may be written

$$Z\{u_n\} = \frac{z}{(z-\alpha)(z-\beta)}.$$

From the quadratic formula,

$$\alpha, \beta = \frac{1 \pm \sqrt{5}}{2}.$$

Using partial fractions,

$$Z\{u_n\} = \frac{1}{\alpha - \beta}\left[\frac{z}{z-\alpha} - \frac{z}{z-\beta}\right].$$

Taking the inverse Z-Transform of this function of z gives

$$\{u_n\} \equiv \left\{\frac{1}{\alpha - \beta}\left[(\alpha)^n - (\beta)^n\right]\right\}.$$

2. Solve the linear difference equation,

$$u_{n+2} - 7u_{n+1} + 10u_n = 16n,$$

given that $u_0 = 6$ and $u_1 = 2$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+1}\} = z.Z\{u_n\} - 6z$$

and

$$Z\{u_{n+2}\} = z^2.Z\{u_n\} - 6z^2 - 2z.$$

Taking the Z-Transform of the
difference equation,

$$z^2.Z\{u_n\} - 6z^2 - 2z - 7\left[z.Z\{u_n\} - 6z\right] + 10Z\{u_n\} = \frac{16z}{(z-1)^2}.$$

On rearrangement,

$$Z\{u_n\}[z^2 - 7z + 10] - 6z^2 + 40z = \frac{16z}{(z-1)^2}.$$

Hence,

$$Z\{u_n\} = \frac{16z}{(z-1)^2(z-5)(z-2)} + \frac{6z^2 - 40z}{(z-5)(z-2)}.$$

Using partial fractions,

$$Z\{u_n\} = z.\left[\frac{4}{z-2} - \frac{3}{z-5} + \frac{4}{(z-1)^2} + \frac{5}{z-1}\right].$$

The solution to the difference equation is therefore

$$\{u_n\} \equiv \{4(2)^n - 3(5)^n + 4n + 5\}.$$

3. Solve the linear difference equation,

$$u_{n+2} + 2u_n = 0,$$

given that $u_0 = 1$ and $u_1 = \sqrt{2}$.

Solution

Using the second shifting theorem,

$$Z\{u_{n+2}\} = z^2 Z\{u_n\} - z^2 - z\sqrt{2}.$$

Taking the Z-Transform of the difference equation,

$$z^2 Z\{u_n\} - z^2 - z\sqrt{2} + 2Z\{u_n\} = 0.$$

On rearrangement,

$$Z\{u_n\} = \frac{z^2 + z\sqrt{2}}{z^2+2} \equiv z.\frac{z+\sqrt{2}}{z^2+2} \equiv z.\frac{z+\sqrt{2}}{(z+j\sqrt{2})(z-j\sqrt{2})}.$$

Using partial fractions,

$$Z\{u_n\} = z\left[\frac{\sqrt{2}(1+j)}{j2\sqrt{2}(z-j\sqrt{2})} + \frac{\sqrt{2}(1-j)}{-j2\sqrt{2}(z+j\sqrt{2})}\right]$$

or

$$Z\{u_n\} \equiv z.\left[\frac{(1-j)}{2(z-j\sqrt{2})} + \frac{(1+j)}{2(z+j\sqrt{2})}\right].$$

Hence,

$$\{u_n\} \equiv \left\{\frac{1}{2}(1-j)(j\sqrt{2})^n + \frac{1}{2}(1+j)(-j\sqrt{2})^n\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[(1-j)(j)^n + (1+j)(-j)^n\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^n\left[\sqrt{2}e^{-j\frac{\pi}{4}}.e^{j\frac{n\pi}{2}} + \sqrt{2}e^{j\frac{\pi}{4}}.e^{-j\frac{n\pi}{2}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}\left[e^{j\frac{(2n-1)\pi}{4}} + e^{-j\frac{(2n-1)\pi}{4}}\right]\right\}$$

$$\equiv \left\{\frac{1}{2}(\sqrt{2})^{n+1}.2\cos\frac{(2n-1)\pi}{4}\right\}$$

$$\equiv \left\{(\sqrt{2})^{n+1}\cos\frac{(2n-1)\pi}{4}\right\}.$$