"JUST THE MATHS"

SLIDES NUMBER

16.1

LAPLACE TRANSFORMS 1
(Definitions and rules)

by

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UNIT 16.1 - LAPLACE TRANSFORMS 1 DEFINITIONS AND RULES

16.1.1 INTRODUCTION

The theory of **"Laplace Transforms"** is used to solve certain kinds of **"differential equation"**.

ILLUSTRATIONS

(a) A **"first order linear differential equation with constant coefficients"**,

$$a\frac{\mathrm{d}x}{\mathrm{d}t} + bx = f(t),$$

together with the value of $x(0)$.

We obtain a formula for x in terms of t which does not include any derivatives.

(b) A **"second order linear differential equation with constant coefficients"**,

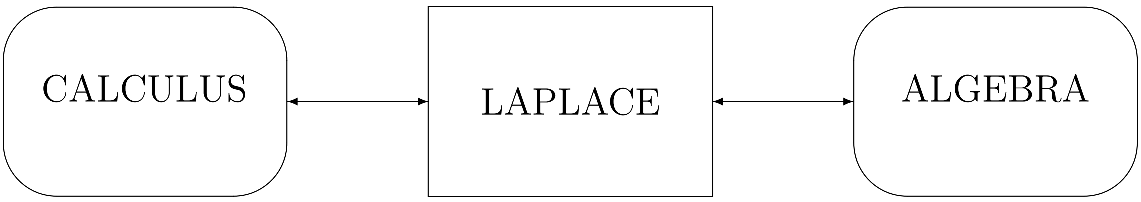
$$a\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + b\frac{\mathrm{d}x}{\mathrm{d}t} + cx = f(t),$$

together with the values of $x(0)$ and $x'(0)$.

We obtain a formula for x in terms of t which does not include any derivatives

The method of Laplace Transforms converts a calculus problem (the differential equation) into an algebra problem (frequently an exercise on partial fractions and/or completing the square).

The solution of the algebra problem is then fed backwards through what is called an **"Inverse Laplace Transform"** and the solution of the differential equation is obtained.



DEFINITION

The Laplace Transform of a given function $f(t)$, defined for $t > 0$, is defined by the definite integral

$$\int_0^\infty e^{-st} f(t)\, \mathrm{d}t,$$

where s is an **arbitrary positive number**.

Notes

(i) The Laplace Transform is usually denoted by $L[f(t)]$ or $F(s)$.

(ii) Although s is an arbitrary positive number, it is occasionally necessary to assume that it is large enough to avoid difficulties in the calculations.

16.1.2 LAPLACE TRANSFORMS OF SIMPLE FUNCTIONS

1. $f(t) \equiv t^n$.

$$F(s) = \int_0^\infty e^{-st} t^n \, \mathrm{d}t = I_n \text{ say.}$$

Hence,

$$I_n = \left[\frac{t^n e^{-st}}{-s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st} t^{n-1} \, \mathrm{d}t = \frac{n}{s}.I_{n-1}.$$

Note:
A **"decaying exponential"** will always have the dominating effect.

We conclude that

$$I_n = \frac{n}{s}.\frac{(n-1)}{s}.\frac{(n-2)}{s}.....\frac{2}{s}.\frac{1}{s}.I_0 = \frac{n!}{s^n}.I_0.$$

But,

$$I_0 = \int_0^\infty e^{-st} \, \mathrm{d}t = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}.$$

Thus,

$$L[t^n] = \frac{n!}{s^{n+1}}.$$

Note:
This result also shows that

$$L[1] = \frac{1}{s},$$

since $1 = t^0$.

2. $f(t) \equiv e^{-at}$.

$$F(s) = \int_0^\infty e^{-st} e^{-at}\ \mathrm{d}t = \int_0^\infty e^{-(s+a)t}\ \mathrm{d}t$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

Hence,

$$L[e^{-at}] = \frac{1}{s+a}.$$

Note:

$$L[e^{bt}] = \frac{1}{s-b}, \quad \text{assuming that} \quad s > b.$$

3. $f(t) \equiv \cos at$.

$$F(s) = \int_0^\infty e^{-st} \cos at\ \mathrm{d}t$$

$$= \left[\frac{e^{-st}\sin at}{a}\right]_0^\infty + \frac{s}{a}\int_0^\infty e^{-st}\sin at\ \mathrm{d}t$$

$$F(s) = 0 + \frac{s}{a}\left\{\left[-\ \frac{e^{-st}\cos at}{a}\right]_0^\infty - \frac{s}{a}\int_0^\infty e^{-st}\cos at\ \mathrm{d}t\right\},$$

which gives

$$F(s) = \frac{s}{a^2} - \frac{s^2}{a^2}.F(s).$$

That is,

$$F(s) = \frac{s}{s^2 + a^2}.$$

In other words,

$$L[\cos at] = \frac{s}{s^2 + a^2}.$$

4. $f(t) \equiv \sin at$.

The method is similar to that for $\cos at$, and we obtain

$$L[\sin at] = \frac{a}{s^2 + a^2}.$$

16.1.3 ELEMENTARY LAPLACE TRANSFORM RULES

1. LINEARITY

If A and B are constants, then

$$L[Af(t) + Bg(t)] = AL[f(t)] + BL[g(t)].$$

Proof:

This follows easily from the linearity of an integral.

EXAMPLE

Determine the Laplace Transform of the function,

$$2t^5 - 7\cos 4t - 1.$$

Solution

$$L[2t^5 + 7\cos 4t - 1] = 2.\frac{5!}{s^6} + 7.\frac{s}{s^2 + 4^2} - \frac{1}{s}$$

$$= \frac{240}{s^6} + \frac{7s}{s^2 + 16} - \frac{1}{s}.$$

2. THE LAPLACE TRANSFORM OF A DERIVATIVE

(a)

$$L[f'(t)] = sL[f(t)] - f(0).$$

Proof:

$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,\mathrm{d}t = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,\mathrm{d}t$

using Integration by Parts.

Thus,

$$L[f'(t)] = -f(0) + sL[f(t)].$$

as required.

(b)

$$L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0).$$

Proof:

Treating $f''(t)$ as the first derivative of $f'(t)$, we have

$$L[f''(t)] = sL[f'(t)] - f'(0).$$

This gives the required result on substituting the expression for $L[f'(t)]$.

Alternative Forms (Using $L[x(t)] = X(s)$)**:**

(i)

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

(ii)

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0) \text{ or } s[sX(s) - x(0)] - x'(0).$$

3. THE (First) SHIFTING THEOREM

$$L[e^{-at}f(t)] = F(s+a).$$

Proof:

$L[e^{-at}f(t)] = \int_0^\infty e^{-st}e^{-at}f(t)\,\mathrm{d}t = \int_0^\infty e^{-(s+a)t}f(t)\,\mathrm{d}t.$

Note:

$$L[e^{bt}f(t)] = F(s-b).$$

EXAMPLE

Determine the Laplace Transform of the function, $e^{-2t}\sin 3t$.

Solution

First, we note that

$$L[\sin 3t] = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}.$$

Replacing s by $(s+2)$, the First Shifting Theorem gives

$$L[e^{-2t}\sin 3t] = \frac{3}{(s+2)^2+9}.$$

4. MULTIPLICATION BY t

$$L[tf(t)] = -\frac{\mathrm{d}}{\mathrm{d}s}[F(s)].$$

Proof:

It may be shown that

$$\frac{\mathrm{d}}{\mathrm{d}s}[F(s)] = \int_0^\infty \frac{\partial}{\partial s}[e^{-st}f(t)]\mathrm{d}t$$

$$= \int_0^\infty -te^{-st}f(t)\ \mathrm{d}t = -\ L[tf(t)].$$

EXAMPLE

Determine the Laplace Transform of the function,

$$t\cos 7t.$$

Solution

$$L[t\cos 7t] = -\frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{s}{s^2+7^2}\right]$$

$$= -\ \frac{(s^2+7^2).1 - s.2s}{(s^2+7^2)^2} = \frac{s^2-49}{(s^2+49)^2}.$$

A TABLE OF LAPLACE TRANSFORMS

$f(t)$	$L[f(t)] = F(s)$
K (a constant)	$\frac{K}{s}$
e^{-at}	$\frac{1}{s+a}$
t^n	$\frac{n!}{s^{n+1}}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\mathrm{sinh} at$	$\frac{a}{s^2-a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\mathrm{cosh} at$	$\frac{s}{s^2-a^2}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t \sin at$	$\frac{2as}{(s^2+a^2)^2}$
$t \cos at$	$\frac{(s^2-a^2)}{(s^2+a^2)^2}$
$\sin at - at \cos at$	$\frac{2a^3}{(s^2+a^2)^2}$

16.1.4 FURTHER LAPLACE TRANSFORM RULES

1.

$$L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = sX(s) - x(0).$$

2.

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s^2X(s) - sx(0) - x'(0)$$

or

$$L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] = s[sX(s) - x(0)] - x'(0).$$

3. **The Initial Value Theorem**

$$\lim_{t\to 0}\ f(t) = \lim_{s\to\infty}\ sF(s),$$

provided that the indicated limits exist.

4. **The Final Value Theorem**

$$\lim_{t\to\infty}\ f(t) = \lim_{s\to 0}\ sF(s),$$

provided that the indicated limits exist.

5. **The Convolution Theorem**

$$L\left[\int_0^t\ f(T)g(t-T)\ \mathrm{d}T\right] = F(s)G(s).$$