“JUST THE MATHS”

SLIDES NUMBER

14.9

PARTIAL DIFFERENTIATION 9
(Taylor’s series)
for
(Functions of several variables)

by

A.J.Hobson

14.9.1 The theory and formula

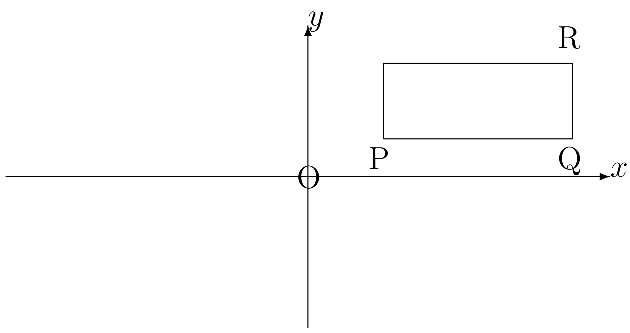
UNIT 14.9
PARTIAL DIFFERENTIATION 9

TAYLOR'S SERIES FOR FUNCTIONS OF SEVERAL VARIABLES

14.9.1 THE THEORY AND FORMULA

First, we obtain a formula for $f(x+h, y+k)$ in terms of $f(x, y)$ and its partial derivatives.

Let P,Q and R denote the points with cartesian co-ordinates, (x, y), $(x+h, y)$ and $(x+h, y+k)$,
respectively.



(a) On the straight line from P to Q, y remains constant, so $f(x, y)$ behaves as a function of x only.

By Taylor's theorem for one independent variable,

$$f(x+h,y) = f(x,y) + f_x(x,y) + \frac{h^2}{2!} f_{xx}(x,y) + \ldots$$

Notes:

(i) $f_x(x,y)$ and $f_{xx}(x,y)$ mean $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$, respectively

(ii) In abbreviated notation,

$$f(\mathrm{Q}) = f(\mathrm{P}) + h f_x(\mathrm{P}) + \frac{h^2}{2!} f_{xx}(\mathrm{P}) + \ ..$$

(b) On the straight line from Q to R, x remains constant, so $f(x,y)$ behaves as a function of y only.

Hence, $f(x+h, y+k) =$

$$f(x+h,y) + k f_x(x+h,y) + \frac{k^2}{2!} f_{xx}(x+h,y) + \ldots$$

Note:

In abbreviated notation,

$$f(\mathrm{R}) = f(\mathrm{Q}) + k f_y(\mathrm{Q}) + \frac{k^2}{2!} f_{yy}(\mathrm{Q}) + ..$$

(c) From the result in (a),

$$f_y(\mathrm{Q}) = f_y(\mathrm{P}) + hf_{yx}(\mathrm{P}) + \frac{h^2}{2!}f_{yxx}(\mathrm{P}) + \ldots$$

and

$$f_{yy}(\mathrm{Q}) = f_{yy}(\mathrm{P}) + hf_{yyx}(\mathrm{P}) + \frac{h^2}{2!}f_{yyxx}(\mathrm{Q}) + \ldots$$

(d) Substituting into (b) gives

$$f(\mathrm{R}) = f(\mathrm{P}) + hf_x(\mathrm{P}) + kf_y(\mathrm{P}) +$$

$$\frac{1}{2!}\left[h^2 f_{xx}(\mathrm{P}) + 2hkf_{yx}(\mathrm{P}) + k^2 f_{yy}(\mathrm{P})\right] + \ldots$$

It may be shown that the complete result can be written as

$$f(x+h, y+k) = f(x,y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) +$$

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f(x,y) + \ \ldots$$

Notes:

(i) The equivalent result for a function of three variables is

$$f(x+h, y+k, z+l) =$$

$$f(x,y,z) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right) f(x,y,z) +$$

$$\frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^2 f(x,y,z) +$$

$$\frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z}\right)^3 f(x,y,z) + \ \ldots$$

(ii) Alternative versions of Taylor's theorem may be obtained by interchanging $x, y, z...$ with $h, k, l....$

For example,

$$f(x+h, y+k) = f(h,k) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(h,k) + \frac{1}{3!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

(iii) Replacing x with $x-h$ and y with $y-k$ in (ii) gives

$$f(x,y) = f(h,k) + \left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right) f(h,k) +$$

$$\frac{1}{2!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^2 f(h,k) +$$

$$\frac{1}{3!}\left((x-h)\frac{\partial}{\partial x} + (y-k)\frac{\partial}{\partial y}\right)^3 f(h,k) + \ \ldots$$

This is the **"Taylor expansion of $f(x,y)$ about the point** (a,b)**"**

(iv) A special case of Taylor's series (for two independent variables) with $h=0$ and $k=0$ is

$$f(x,y) =$$

$$f(0,0) + \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f(0,0) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f(0,0) + \ldots$$

This is called a **"MacLaurin's series"**

EXAMPLE

Determine the Taylor series expansion of the function $f\left(x+1, y+\frac{\pi}{3}\right)$ in ascending powers of x and y when

$$f(x,y) \equiv \sin xy,$$

neglecting terms of degree higher than two.

Solution

$$f\left(x+1, y+\frac{\pi}{3}\right) = f\left(1, \frac{\pi}{3}\right) +$$

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) f\left(1, \frac{\pi}{3}\right) + \frac{1}{2!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f\left(1, \frac{\pi}{3}\right) + \ldots$$

The first term on the right has value $\sqrt{3}/2$.

The partial derivatives required are as follows:

$$\frac{\partial f}{\partial x} \equiv y\cos xy = -\frac{\pi}{6} \quad \text{at} \quad x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial f}{\partial y} \equiv x\cos xy = \frac{1}{2} \quad \text{at} \quad x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x^2} \equiv -y^2\sin xy = -\frac{\pi^2\sqrt{3}}{18} \quad \text{at} \quad x=1, \quad y=\frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \cos xy - xy \sin xy = \frac{1}{2} - \frac{\pi\sqrt{3}}{6} \text{ at } x = 1, \; y = \frac{\pi}{3};$$

$$\frac{\partial^2 f}{\partial y^2} \equiv -x^2 \sin xy = -\frac{\sqrt{3}}{2} \text{ at } x = 1, \quad y = \frac{\pi}{3}.$$

Neglecting terms of degree higher than two,

$$\sin xy =$$

$$\frac{\sqrt{3}}{2} + \frac{\pi}{6}x + \frac{1}{2}y - \frac{\sqrt{3}\pi^2}{36}x^2 + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{6}\right) xy - \frac{\sqrt{3}}{4}y^2 + \ldots$$