"JUST THE MATHS"

SLIDES NUMBER

14.8

PARTIAL DIFFERENTIATION 8
(Dependent and independent functions)

by

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UNIT 14.8
PARTIAL DIFFERENTIATION 8

DEPENDENT AND INDEPENDENT FUNCTIONS

14.8.1 THE JACOBIAN

Suppose that

$$u \equiv u(x, y) \quad \text{and} \quad v \equiv v(x, y)$$

are two functions of two independent variables,
x and y.

Then, it is not normally possible to express u solely in terms of v, nor v solely in terms of u.

However, it may sometimes be possible

ILLUSTRATIONS

1. If

$$u \equiv \frac{x+y}{x} \quad \text{and} \quad v \equiv \frac{x-y}{y},$$

then,

$$u \equiv 1 + \frac{x}{y} \quad \text{and} \quad v \equiv \frac{x}{y} - 1.$$

This gives

$$(u-1)(v+1) \equiv \frac{x}{y}.\frac{y}{x} \equiv 1.$$

Hence,

$$u \equiv 1 + \frac{1}{v+1} \quad \text{and} \quad v \equiv \frac{1}{u-1} - 1.$$

2. If

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2,$$

then,

$$v \equiv u^2 \quad \text{and} \quad u \equiv \pm\sqrt{v}.$$

If u and v are **not** connected by an identical relationship, they are said to be **"independent functions"**

THEOREM

Two functions, $u(x, y)$ and $v(x, y)$ are independent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \not\equiv 0.$$

Proof:

We prove that $u(x,y)$ and $v(x,y)$ are dependent if and only if

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

(a) Suppose that

$$v \equiv v(u).$$

Then,

$$\frac{\partial v}{\partial x} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}.\frac{\partial u}{\partial y}.$$

Thus,

$$\frac{\partial v}{\partial x} \div \frac{\partial u}{\partial x} \equiv \frac{\partial v}{\partial y} \div \frac{\partial u}{\partial y} \equiv \frac{\mathrm{d}v}{\mathrm{d}u}$$

or

$$\frac{\partial u}{\partial x}.\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}.\frac{\partial u}{\partial y} \equiv 0.$$

That is,

$$J \equiv 0.$$

(b) Secondly, suppose that

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \equiv 0.$$

In theory, we could express v in terms of u and x by eliminating y between $u(x, y)$ and $v(x, y)$.

We assume that

$$v \equiv A(u, x)$$

and show that $A(u, x)$ does not contain x.

We have

$$\left(\frac{\partial v}{\partial x}\right)_y \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u$$

and

$$\left(\frac{\partial v}{\partial y}\right)_x \equiv \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x .$$

Hence, if $J \equiv 0$,

$$\begin{vmatrix} \left(\frac{\partial u}{\partial x}\right)_y & \left(\frac{\partial u}{\partial y}\right)_x \\ \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_y + \left(\frac{\partial A}{\partial x}\right)_u & \left(\frac{\partial A}{\partial u}\right)_x \cdot \left(\frac{\partial u}{\partial y}\right)_x \end{vmatrix} \equiv 0.$$

On expansion, this gives

$$\left(\frac{\partial u}{\partial y}\right)_x \cdot \left(\frac{\partial A}{\partial x}\right)_u \equiv 0.$$

If the first of these two is equal to zero, then u contains only x.

Hence, x could be expressed in terms of u giving v as a function of u only.

If the second is equal to zero, then A contains no x's and, again, v is a function of u only.

Notes:

(i) The determinant

$$J \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

may also be denoted by

$$\frac{\partial(u,v)}{\partial(x,y)}.$$

It is called the **"Jacobian determinant"** or simply the **"Jacobian"** of u and v with respect to x and y.

(ii) Similar Jacobian determinants may be used to test for the dependence or independence of three functions of three variables, four functions of four variables, and so on.

For example, the three functions

$$u \equiv u(x, y, z), \quad v \equiv v(x, y, z) \text{ and } w \equiv w(x, y, z)$$

are independent if and only if

$$J \equiv \frac{\partial(u, v, w)}{\partial(x, y, z)} \equiv \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \not\equiv 0.$$

ILLUSTRATIONS

1.

$$u \equiv \frac{x + y}{x} \text{ and } v \equiv \frac{x - y}{y}$$

are **not** independent, since

$$J \equiv \begin{vmatrix} -\frac{y}{x^2} & \frac{1}{x} \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} \equiv \frac{1}{xy} - \frac{1}{xy} \equiv 0.$$

2.

$$u \equiv x + y \quad \text{and} \quad v \equiv x^2 + 2xy + y^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 1 & 1 \\ 2x + 2y & 2x + 2y \end{vmatrix} \equiv 0.$$

3.

$$u \equiv x^2 + 2y \quad \text{and} \quad v \equiv xy$$

are independent, since

$$J \equiv \begin{vmatrix} 2x & 2 \\ y & x \end{vmatrix} \equiv 2x^2 - 2y \not\equiv 0.$$

4.

$$u \equiv x^2 - 2y + z, \quad v \equiv x + 3y^2 - 2z, \quad \text{and} \quad w \equiv 5x + y + z^2$$

are **not** independent, since

$$J \equiv \begin{vmatrix} 2x & -2 & 1 \\ 1 & 6y & -2 \\ 5 & 1 & 2z \end{vmatrix} \equiv 24xyz + 4x - 30y + 4z + 25 \not\equiv 0.$$