

“JUST THE MATHS”

SLIDES NUMBER

14.7

**PARTIAL DIFFERENTIATION 7
(Change of independent variable)**

by

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14.7.1 Illustrations of the method

UNIT 14.7

PARTIAL DIFFERENTIATION 7

CHANGE OF INDEPENDENT VARIABLE

14.7.1 ILLUSTRATIONS OF THE METHOD

The following technique would be necessary, for example, in changing from one geometrical reference system to another, especially with

“partial differential equations”.

The method is an application of the chain rule for partial derivatives and is illustrated with examples.

EXAMPLES

1. Express, in plane polar co-ordinates, r and θ , the following partial differential equations:

(a)

$$\frac{\partial V}{\partial x} + 5 \frac{\partial V}{\partial y} = 1;$$

(b)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

Solution

Both differential equations involve a function, $V(x, y)$, where

$$x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.$$

Hence,

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial r},$$

or

$$\frac{\partial V}{\partial r} = \frac{\partial V}{\partial x} \cos \theta + \frac{\partial V}{\partial y} \sin \theta$$

and

$$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y} \cdot \frac{\partial y}{\partial \theta},$$

or

$$\frac{\partial V}{\partial \theta} = -\frac{\partial V}{\partial x} r \sin \theta + \frac{\partial V}{\partial y} r \cos \theta.$$

We eliminate, first $\frac{\partial V}{\partial y}$, and then $\frac{\partial V}{\partial x}$ to obtain

$$\frac{\partial V}{\partial x} = \cos \theta \frac{\partial V}{\partial r} - \frac{\sin \theta}{r} \frac{\partial V}{\partial \theta}$$

and

$$\frac{\partial V}{\partial y} = \sin \theta \frac{\partial V}{\partial r} + \frac{\cos \theta}{r} \frac{\partial V}{\partial \theta}.$$

Hence, the differential equation, (a), becomes

$$(\cos \theta + 5 \sin \theta) \frac{\partial V}{\partial r} + \left(\frac{5 \cos \theta}{r} - \sin \theta \right) \frac{\partial V}{\partial \theta} = 1.$$

To find the second-order derivatives of V with respect to x and y , we write the formulae for the first-order derivatives in the form

$$\frac{\partial}{\partial x}[V] = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) [V]$$

and

$$\frac{\partial}{\partial y}[V] = \left(\sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \right) [V].$$

From these,

$$\frac{\partial^2 V}{\partial x^2} = \left(\cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \right) \left(\cos \theta \frac{\partial V}{\partial r} - \frac{\sin \theta}{r} \frac{\partial V}{\partial \theta} \right),$$

which gives

$$\begin{aligned} \frac{\partial^2 V}{\partial x^2} &= \cos^2 \theta \frac{\partial^2 V}{\partial r^2} + \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial V}{\partial \theta} \\ &\quad - \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 V}{\partial r \partial \theta} + \frac{\sin^2 \theta}{r} \frac{\partial V}{\partial r} + \frac{\sin^2 \theta}{r^2} \frac{\partial^2 V}{\partial \theta^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2 V}{\partial y^2} &= \sin^2 \theta \frac{\partial^2 V}{\partial r^2} - \frac{2 \sin \theta \cos \theta}{r^2} \frac{\partial V}{\partial \theta} \\ &+ \frac{2 \sin \theta \cos \theta}{r} \frac{\partial^2 V}{\partial r \partial \theta} + \frac{\cos^2 \theta}{r} \frac{\partial V}{\partial r} + \frac{\cos^2 \theta}{r^2} \frac{\partial^2 V}{\partial \theta^2}. \end{aligned}$$

Adding these together gives the differential equation, (b), in the form

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} = 0.$$

2. Express the differential equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

(a) in cylindrical polar co-ordinates

and

(b) in spherical polar co-ordinates.

Solution

(a) Using

$$x = r \cos \theta, \quad y = r \sin \theta \quad \text{and} \quad z = z,$$

we may use the results of the previous example to give

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{1}{r^2} \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

(b) Using

$$x = u \sin \phi \cos \theta, \quad y = u \sin \phi \sin \theta, \quad \text{and} \quad z = u \cos \phi,$$

we could write out three formulae for $\frac{\partial V}{\partial u}$, $\frac{\partial V}{\partial \theta}$ and $\frac{\partial V}{\partial \phi}$ and then solve for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; but this is complicated.

However, the result in part (a) provides a shorter method as follows:

Cylindrical polar co-ordinates are expressible in terms of spherical polar co-ordinates by the formulae

$$z = u \cos \phi, \quad r = u \sin \phi, \quad \theta = \theta.$$

Hence, by using the previous example with z , r , θ in place of x , y , z , respectively, and u , ϕ in place of r , θ , respectively, we obtain

$$\frac{\partial^2 V}{\partial z^2} + \frac{\partial^2 V}{\partial r^2} = \frac{\partial^2 V}{\partial u^2} + \frac{1}{u} \frac{\partial V}{\partial u} + \frac{1}{u^2} \frac{\partial^2 V}{\partial \phi^2}.$$

Therefore, to complete the conversion we need only to consider $\frac{\partial V}{\partial r}$.

By using r , u , ϕ in place of y , r , θ , respectively, the previous formula for $\frac{\partial V}{\partial y}$ gives

$$\frac{\partial V}{\partial r} = \sin \phi \frac{\partial V}{\partial u} + \frac{\cos \phi}{u} \frac{\partial V}{\partial \phi}.$$

The given differential equation thus becomes

$$\frac{\partial^2 V}{\partial u^2} + \frac{1}{u} \frac{\partial V}{\partial u} + \frac{1}{u^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1}{u^2 \sin^2 \phi} \frac{\partial^2 V}{\partial \theta^2} + \frac{1}{u \sin \phi} \left[\sin \phi \frac{\partial V}{\partial u} + \frac{\cos \phi}{u} \frac{\partial V}{\partial \phi} \right] = 0.$$

That is,

$$\frac{\partial^2 V}{\partial u^2} + \frac{2}{u} \frac{\partial V}{\partial u} + \frac{1}{u^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\cot \phi}{u^2} \frac{\partial V}{\partial \phi} + \frac{1}{u^2 \sin^2 \phi} \frac{\partial^2 V}{\partial \theta^2} = 0.$$