"JUST THE MATHS"

SLIDES NUMBER

14.6

PARTIAL DIFFERENTIATION 6
(Implicit functions)

by

A.J.Hobson

UNIT 14.6
PARTIAL DIFFERENTIATION 6

IMPLICIT FUNCTIONS

14.6.1 FUNCTIONS OF TWO VARIABLES

The chain rule can be applied to implicit relationships of the form,

$$f(x, y) = \text{constant},$$

between two independent variables, x and y.

We may determine the total derivative of y with respect to x.

Explanation

Taking x as the single independent variable, $f(x, y)$ is a function of x and y in which both x and y are functions of x.

Differentiating $f(x, y) =$ constant with respect to x gives

$$\frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

In other words,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}x} = 0.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}.$$

EXAMPLES

1. If

$$f(x, y) \equiv x^3 + 4x^2y - 3xy + y^2 = 0,$$

determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 8xy - 3y \ \text{ and } \ \frac{\partial f}{\partial y} = 4x^2 - 3x + 2y.$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 + 8xy - 3y}{4x^2 - 3x + 2y}.$$

2. If

$$f(x, y) \equiv x\sin(2x - 3y) + y\cos(2x - 3y),$$

determine an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution

$$\frac{\partial f}{\partial x} = \sin(2x-3y)+2x\cos(2x-3y)-2y\sin(2x-3y)$$

and

$$\frac{\partial f}{\partial y} = -3x\cos(2x-3y)+\cos(2x-3y)+3y\sin(2x-3y).$$

Hence,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin(2x - 3y) + 2x\cos(2x - 3y) - 2y\sin(2x - 3y)}{3x\cos(2x - 3y) - \cos(2x - 3y) - 3y\sin(2x - 3y)}.$$

14.6.2 FUNCTIONS OF THREE VARIABLES

For relationships of the form,

$$f(x, y, z) = \text{constant},$$

let x and y be independent of each other.

Then, $f(x, y, z)$ is a function of x, y and z, where x, y and z are **all** functions of x and y.

The chain rule gives

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial f}{\partial y}.\frac{\partial y}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0.$$

But,

$$\frac{\partial x}{\partial x}=1 \text{ and } \frac{\partial y}{\partial x}=0.$$

Hence,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}.\frac{\partial z}{\partial x}=0,$$

giving

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

and, similarly,

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

EXAMPLES

1. If

$$f(x,y,z) \equiv z^2xy + zy^2x + x^2 + y^2 = 5,$$

determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = z^2y + zy^2 + 2x,$$

$$\frac{\partial f}{\partial y} = z^2x + 2zyx + 2y$$

and

$$\frac{\partial f}{\partial z} = 2zxy + y^2x.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{z^2y + zy^2 + 2x}{2zxy + y^2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{z^2x + 2zyx + 2y}{2zxy + y^2x}.$$

2. If

$$f(x,y,z) \equiv xe^{y^2+2z},$$

determine expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution

$$\frac{\partial f}{\partial x} = e^{y^2+2z},$$

$$\frac{\partial f}{\partial y} = 2yxe^{y^2+2z},$$

and

$$\frac{\partial f}{\partial z} = 2xe^{y^2+2z}.$$

Hence,

$$\frac{\partial z}{\partial x} = -\frac{e^{y^2+2z}}{2xe^{y^2+2z}} = -\frac{1}{2x}$$

and

$$\frac{\partial z}{\partial y} = -\frac{2yxe^{y^2+2z}}{2xe^{y^2+2z}} = -y.$$