"JUST THE MATHS"

SLIDES NUMBER

14.5

PARTIAL DIFFERENTIATION 5
(Partial derivatives of composite functions)

by

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UNIT 14.5
PARTIAL DIFFERENTIATION 5

PARTIAL DERIVATIVES OF COMPOSITE FUNCTIONS

14.5.1 SINGLE INDEPENDENT VARIABLES

We shall be concerned with functions, $f(x, y...)$, of two or more variables in which those variables are not independent, but are themselves dependent on some other variable, t.

The problem is to calculate the rate of increase (positive or negative) of such functions with respect to t.

Let t be subject to a small increment of δt, so that the variables, $x, y...$, are subject to small increments of $\delta x,\ \delta y,\ \ldots$, respectively.

The corresponding increment, δf, in $f(x, y...)$, is given by

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \ldots$$

Note: It is not essential to use a specific **formula**, such as $w = f(x, y..)$.

Dividing throughout by δt gives

$$\frac{\delta f}{\delta t} \simeq \frac{\partial f}{\partial x}.\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}.\frac{\delta y}{\delta t} + \ldots$$

Allowing δt to tend to zero, we obtain the standard result for the **"total derivative"** of $f(x, y..)$ with respect to t,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \ldots$$

This rule may be referred to as the **"chain rule"**, but more advanced versions of it will appear later.

EXAMPLES

1. A point, P, is moving along the curve of intersection of the surface whose cartesian equation is

$$\frac{x^2}{16} - \frac{y^2}{9} = z \quad \text{(a} \quad \text{Paraboloid)}$$

and the surface whose cartesian equation is

$$x^2 + y^2 = 5 \quad \text{(a} \;\; \text{Cylinder)}.$$

If x is increasing at 0.2 cms/sec, how fast is z changing when $x = 2$?

Solution

We may use the formula

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial z}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial z}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t},$$

where

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}.\frac{\mathrm{d}x}{\mathrm{d}t} = 0.2\frac{\mathrm{d}y}{\mathrm{d}x}.$$

From the equation of the paraboloid,

$$\frac{\partial z}{\partial x} = \frac{x}{8} \text{ and } \frac{\partial z}{\partial y} = -\frac{2y}{9}.$$

From the equation of the cylinder,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}.$$

Substituting $x = 2$ gives $y = \pm 1$ on the curve of intersection, so that

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \left(\frac{2}{8}\right)(0.2) + \left(-\frac{2}{9}\right)(\pm 1)(0.2)\left(\frac{-2}{\pm 1}\right) = 0.2\left(\frac{1}{4} + \frac{4}{9}\right)$$

$$= \frac{5}{36} \text{ cms/sec.}$$

2. Determine the total derivative of u with respect to t in the case when

$u = xy + yz + zx, \;\; x = e^t, \;\; y = e^{-t}$ and $z = x + y.$

Solution

We use the formula

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{\partial u}{\partial x}.\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial u}{\partial y}.\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial u}{\partial z}.\frac{\mathrm{d}z}{\mathrm{d}t},$$

where

$$\frac{\partial u}{\partial x} = y + z, \quad \frac{\partial u}{\partial y} = z + x, \quad \frac{\partial u}{\partial z} = x + y$$

and

$$\frac{\mathrm{d}x}{\mathrm{d}t} = e^t = x, \;\; \frac{\mathrm{d}y}{\mathrm{d}t} = -e^{-t} = -y, \;\; \frac{\mathrm{d}z}{\mathrm{d}t} = e^t - e^{-t} = x - y.$$

Hence,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (y + z)x - (z + x)y + (x + y)(x - y)$$

$$= -zy + zx + x^2 - y^2$$

$$= z(x - y) + (x - y)(x + y).$$

That is,

$$\frac{\mathrm{d}u}{\mathrm{d}t} = (x - y)(x + y + z).$$

14.5.2 SEVERAL INDEPENDENT VARIABLES

We may now extend the previous work to functions, $f(x, y..)$, of two or more variables in which $x, y..$ are each dependent on two or more variables, $s, t..$

Since the function $f(x, y..)$ is dependent on $s, t..$, we may wish to determine its **partial** derivatives with respect to any one of these (independent) variables.

The result previously established for a **single** independent variable may easily be adapted as follows:

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial s} + \ldots$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}.\frac{\partial y}{\partial t} + \ldots$$

Again, this is referred to as the **"chain rule"**.

EXAMPLES

1. Determine the first-order partial derivatives of z with respect to r and θ in the case when

$$z = x^2 + y^2, \quad \text{where} \quad x = r\cos\theta \quad \text{and} \quad y = r\sin 2\theta.$$

Solution

We may use the formulae

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial r},$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial \theta}.$$

These give

$$\frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin 2\theta$$

$$= 2r\left(\cos^2\theta + \sin^2 2\theta\right).$$

$$\frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2y(2r\cos 2\theta)$$

$$= 2r^2\left(2\cos 2\theta\sin 2\theta - \cos\theta\sin\theta\right).$$

2. Determine the first-order partial derivatives of w with respect to u, θ and ϕ in the case when

$$w = x^2 + 2y^2 + 2z^2,$$

where

$$x = u\sin\phi\cos\theta, \quad y = u\sin\phi\sin\theta \quad \text{and} \quad z = u\cos\phi.$$

Solution

$$\frac{\partial w}{\partial u}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial u}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial u}+\frac{\partial w}{\partial z}.\frac{\partial z}{\partial u},$$

$$\frac{\partial w}{\partial \theta}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial \theta}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial \theta}+\frac{\partial w}{\partial z}.\frac{\partial z}{\partial \theta}$$

$$\frac{\partial w}{\partial \phi}=\frac{\partial w}{\partial x}.\frac{\partial x}{\partial \phi}+\frac{\partial w}{\partial y}.\frac{\partial y}{\partial \phi}+\frac{\partial w}{\partial z}.\frac{\partial z}{\partial \phi}$$

These give

$$\frac{\partial w}{\partial u}=2x\sin\phi\cos\theta+4y\sin\phi\sin\theta+4z\cos\phi$$

$$=2u\sin^2\phi\cos^2\theta+4u\sin^2\phi\sin^2\theta+4u\cos^2\phi,$$

$$\frac{\partial w}{\partial \theta}=-2xu\sin\phi\sin\theta+4yu\sin\phi\cos\theta$$

$$=-2u^2\sin^2\phi\sin\theta\cos\theta+4u^2\sin^2\phi\sin\theta\cos\theta$$

$$=2u^2\sin^2\phi\sin\theta\cos\theta$$

$$\frac{\partial w}{\partial \phi}=2xu\cos\phi\cos\theta+4yu\cos\phi\sin\theta-4zu\sin\phi$$

$$=2u^2\sin\phi\cos\phi\cos^2\theta+4u^2\sin\phi\cos\phi\sin^2\theta$$

$$-4u^2\sin\phi\cos\phi$$

$$=2u^2\sin\phi\cos\phi\left(\cos^2\theta+2\sin^2\theta-2\right).$$