

“JUST THE MATHS”

SLIDES NUMBER

14.4

**PARTIAL DIFFERENTIATION 4
(Exact differentials)**

by

A.J.Hobson

**14.4.1 Total differentials
14.4.2 Testing for exact differentials
14.4.3 Integration of exact differentials**

UNIT 14.4

PARTIAL DIFFERENTIATION 4

EXACT DIFFERENTIALS

14.4.1 TOTAL DIFFERENTIALS

The expression,

$$\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \dots,$$

is an approximation for the increment (or error), δf in the function $f(x, y, \dots)$ when x, y etc. are subject to increments (or errors) of $\delta x, \delta y$ etc., respectively

The expression may be called the “**total differential**” of $f(x, y, \dots)$, and may be denoted by df , giving

$$df \simeq \delta f.$$

OBSERVATIONS

Consider the formula,

$$df = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \dots$$

(a) If $f(x, y, \dots) \equiv x$, then $df = \delta x$.

Hence,

$$dx = \delta x.$$

(b) If $f(x, y, \dots) \equiv y$, then $df = \delta y$.

Hence,

$$dy = \delta y.$$

(c) The total differential of each **independent** variable is the same as the small increment (or error) in that variable;

but the total differential of the **dependent** variable is only approximately equal to the increment (or error) in that variable.

(d) The observations may be summarised by the formula,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \dots$$

14.4.2 TESTING FOR EXACT DIFFERENTIALS

In general, an expression of the form,

$$P(x, y, \dots)dx + Q(x, y, \dots)dy + \dots,$$

will not be the total differential of a function $f(x, y, \dots)$ unless $P(x, y, \dots)$, $Q(x, y, \dots)$ etc. can be identified with $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ etc., respectively.

If this IS possible, then the expression is known as an “**exact differential**”.

RESULTS

(i) The expression,

$$P(x, y)dx + Q(x, y)dy,$$

is an exact differential if and only if

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

Proof:

(a) If the expression,

$$P(x, y)dx + Q(x, y)dy,$$

is an exact differential, df , then

$$\frac{\partial f}{\partial x} \equiv P(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y} \equiv Q(x, y).$$

Hence,

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x} \left(\equiv \frac{\partial^2 f}{\partial x \partial y} \right).$$

(b) Conversely, suppose that

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}.$$

We can certainly say that

$$P(x, y) \equiv \frac{\partial u}{\partial x}$$

for some function $u(x, y)$, since $P(x, y)$ could be integrated partially with respect to x .

But then,

$$\frac{\partial Q}{\partial x} \equiv \frac{\partial P}{\partial y} \equiv \frac{\partial^2 u}{\partial y \partial x}.$$

On integrating partially with respect to x ,

$$Q(x, y) = \frac{\partial u}{\partial y} + A(y),$$

where $A(y)$ is an **arbitrary** function of y .

Thus,

$$P(x, y)dx + Q(x, y)dy = \frac{\partial u}{\partial x}dx + \left(\frac{\partial u}{\partial y} + A(y) \right) dy.$$

The right-hand side is the exact differential of

$$u(x, y) + \int A(y) dy.$$

(ii) By similar reasoning, it may be shown that

$$P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz$$

is an exact differential when

$$\frac{\partial P}{\partial y} \equiv \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \text{and} \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}.$$

ILLUSTRATIONS

1.

$$x dx + y dy = d \left[\frac{1}{2} (x^2 + y^2) \right].$$

2.

$$y dx + x dy = d[xy].$$

3.

$$y dx - x dy$$

is not an exact differential, since

$$\frac{\partial y}{\partial y} = 1 \quad \text{and} \quad \frac{\partial(-x)}{\partial x} = -1.$$

4.

$$2 \ln y dx + (x + z) dy + z^2 dz$$

is not an exact differential, since

$$\frac{\partial(2 \ln y)}{\partial y} = \frac{2}{y}, \quad \text{and} \quad \frac{\partial(x + z)}{\partial x} = 1.$$

14.4.3 INTEGRATION OF EXACT DIFFERENTIALS

The method may be illustrated by the following examples:

EXAMPLES

1. Verify that the expression,

$$(x + y \cos x)dx + (1 + \sin x)dy,$$

is an exact differential and obtain the function of which it is the total differential

Solution

Firstly,

$$\frac{\partial}{\partial y}(x + y \cos x) \equiv \frac{\partial}{\partial x}(1 + \sin x) \equiv \cos x$$

and, hence, the expression is an exact differential.

Secondly, suppose that the expression is the total differential of the function, $f(x, y)$.

Then,

$$\frac{\partial f}{\partial x} \equiv x + y \cos x \quad \text{--- --- --- --- --- --- --- --- --- (1)}$$

and

$$\frac{\partial f}{\partial y} \equiv 1 + \sin x. \quad \text{--- --- --- --- --- --- --- --- --- (2)}$$

Integrating (1) partially with respect to x gives

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + A(y),$$

where $A(y)$ is an **arbitrary** function of y only.

Substituting this result into (2) gives

$$\sin x + \frac{dA}{dy} \equiv 1 + \sin x.$$

That is,

$$\frac{dA}{dy} \equiv 1,$$

Hence,

$$A(y) \equiv y + \text{constant}.$$

We conclude that

$$f(x, y) \equiv \frac{x^2}{2} + y \sin x + y + \text{constant}.$$

2. Verify that the expression,

$$(yz + 2)dx + (xz + 6y)dy + (xy + 3z^2)dz,$$

is an exact differential and obtain the function of which it is the total differential.

Solution

Firstly,

$$\frac{\partial}{\partial y}(yz + 2) \equiv \frac{\partial}{\partial x}(xz + 6y) \equiv z,$$

$$\frac{\partial}{\partial z}(xz + 6y) \equiv \frac{\partial}{\partial y}(xy + 3z^2) \equiv x,$$

and

$$\frac{\partial}{\partial x}(xy + 3z^2) \equiv \frac{\partial}{\partial z}(yz + 2) \equiv y;$$

so that the given expression is an exact differential.

Suppose it is the total differential of the function, $F(x, y, z)$.

Then,

$$\frac{\partial F}{\partial x} \equiv yz + 2, \quad \text{--- --- --- --- --- (1)}$$

$$\frac{\partial F}{\partial y} \equiv xz + 6y, \quad \text{--- --- --- --- --- (2)}$$

$$\frac{\partial F}{\partial z} \equiv xy + 3z^2. \quad \text{--- --- --- --- --- (3)}$$

Integrating (1) partially with respect to x gives

$$F(x, y, z) \equiv xyz + 2x + A(y, z),$$

where $A(y, z)$ is an arbitrary function of y and z only.

Substituting into both (2) and (3) gives

$$\begin{aligned}xz + \frac{\partial A}{\partial y} &\equiv xz + 6y, \\xy + \frac{\partial A}{\partial z} &\equiv xy + 3z^2.\end{aligned}$$

That is,

$$\frac{\partial A}{\partial y} \equiv 6y, \quad \text{--- --- --- --- --- (4)}$$

$$\frac{\partial A}{\partial z} \equiv 3z^2. \quad \text{--- --- --- --- --- (5)}$$

Integrating (4) partially with respect to y gives

$$A(y, z) \equiv 3y^2 + B(z),$$

where $B(z)$ is an arbitrary function of z only.

Substituting into (5) gives

$$\frac{dB}{dz} \equiv 3z^2,$$

which implies that $B(z) \equiv z^3 + \text{constant}$ and, hence,

$$F(x, y, z) \equiv xyz + 2x + 3y^2 + z^3 + \text{constant}.$$