"JUST THE MATHS"

SLIDES NUMBER

14.3

PARTIAL DIFFERENTIATION 3
(Small increments and small errors)

by

A.J.Hobson

UNIT 14.3
PARTIAL DIFFERENTIATION 3

SMALL INCREMENTS AND SMALL ERRORS

14.3.1 FUNCTIONS OF ONE INDEPENDENT VARIABLE - A RECAP

If

$$y = f(x),$$

then

(a) The **increment**, δy, in y, due to an increment of δx, in x is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

(b) The **error**, δy, in y, due to an error of δx in x, is given (to the first order of approximation) by

$$\delta y \simeq \frac{\mathrm{d}y}{\mathrm{d}x}\delta x.$$

14.3.2 FUNCTIONS OF MORE THAN ONE INDEPENDENT VARIABLE

Let

$$z = f(x, y),$$

where x and y are independent variables.

If x is subject to a small increment (or a small error) of δx, while y remains constant, then the corrresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x.$$

Similarly, if y is subject to a small increment (or a small error) of δy, while x remains constant, then the corresponding increment (or error) of δz in z will be given approximately by

$$\delta z \simeq \frac{\partial z}{\partial y}\delta y.$$

It can be shown that, for increments (or errors) in both x and y,

$$\delta z \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y.$$

Notes:

(i) From **"Taylor's Theorem"**

$$f(x+\delta x, y+\delta y) = f(x,y) + \left(\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right)$$

$$+ \left(\frac{\partial^2 z}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 z}{\partial x \partial y}\delta x \delta y + \frac{\partial^2 z}{\partial y^2}(\delta y)^2\right) + \ldots,$$

which shows that

$$\delta z = f(x+\delta x, y+\delta y) - f(x,y) \simeq \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y,$$

to the first order of approximation.

(ii) The formula for a function of two independent variables may be extended to functions of a greater number of independent variables.

For example, if

$$w = F(x, y, z),$$

then

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

EXAMPLES

1. A rectangle has sides of length $x$cms. and $y$cms. Calculate, approximately, in terms of x and y, the increment in the area, A, of the rectangle when x and y are subject to increments of δx and δy respectively.

   Solution

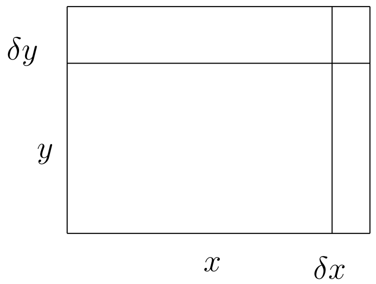
   The area, A, is given by

   $$A = xy,$$

   so that

   $$\delta A \simeq \frac{\partial A}{\partial x}\delta x + \frac{\partial A}{\partial y}\delta y = y\delta x + x\delta y.$$

Note:
The exact value of δA may be seen in the following diagram:



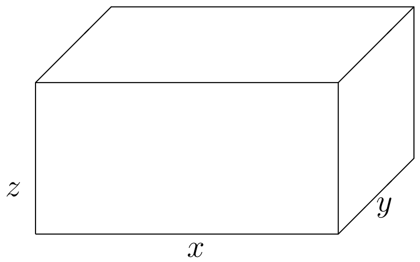
The difference between the approximate value and the exact value is represented by the area of the small rectangle having sides δxcms. and δycms.

2. In measuring a rectangular block of wood, the dimensions were found to be 10cms., 12cms and 20cms. with a possible error of $\pm 0.05 cms.$ in each.

   Calculate, approximately, the greatest possible error in the surface area, S, of the block and the percentage error so caused.

Solution

First, denote the lengths of the edges of the block by x, y and z.



The surface area, S, is given by

$$S = 2(xy + yz + zx),$$

which has the value 1120cms^2 when $x = 10\text{cms.}$, $y = 12\text{cms.}$ and $z = 20\text{cms.}$

Also,

$$\delta S \simeq \frac{\partial S}{\partial x}\delta x + \frac{\partial S}{\partial y}\delta y + \frac{\partial S}{\partial z}\delta z,$$

which gives

$$\delta S \simeq 2(y + z)\delta x + 2(x + z)\delta y + 2(y + x)\delta z.$$

On substituting $x = 10$, $y = 12$, $z = 20$, $\delta x = \pm 0.05$, $\delta y = \pm 0.05$ and $\delta z = \pm 0.05$, we obtain

$$\delta S \simeq \pm 2(12{+}20)(0.05) \pm 2(10{+}20)(0.05) \pm 2(12{+}10)(0.05).$$

The greatest error will occur when all the terms of the above expression have the same sign.

Hence, the greatest error is given by

$$\delta S_{\max} \simeq \pm 8.4\text{cms.}^2$$

This represents a percentage error of approximately

$$\pm\frac{8.4}{1120} \times 100 = \pm 0.75$$

3. If

$$w = \frac{x^3 z}{y^4},$$

calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

We have

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y + \frac{\partial w}{\partial z}\delta z.$$

That is,

$$\delta w \simeq \frac{3x^2 z}{y^4}\delta x - \frac{4x^3 z}{y^5}\delta y + \frac{x^3}{y^4}\delta z,$$

where

$$\delta x = -\frac{3x}{100}, \quad \delta y = \frac{y}{100} \quad \text{and} \quad \delta z = \frac{2z}{100}.$$

Thus,

$$\delta w \simeq \frac{x^3 z}{y^4}\left[-\frac{9}{100} - \frac{4}{100} + \frac{2}{100}\right] = -\frac{11w}{100}.$$

The percentage error in w is given approximately by

$$\frac{\delta w}{w} \times 100 = -11.$$

That is, w is too small by approximately 11%.

14.3.3 THE LOGARITHMIC METHOD

For percentage errors or percentage increments, we may use logarithms if the right hand side of the formula involves a product, a quotient, or a combination of these two in which the independent variables are separated.

A formula of this type would be,

$$w = \frac{x^3 z}{y^4}.$$

The method is to take the natural logarithms of both sides of the equation before considering any partial derivatives.

GENERAL THEORY (two variables)

Suppose that

$$z = f(x, y).$$

Then,

$$\ln z = \ln f(x, y).$$

If we let $w = \ln z$, then

$$w = \ln f(x, y),$$

giving

$$\delta w \simeq \frac{\partial w}{\partial x}\delta x + \frac{\partial w}{\partial y}\delta y.$$

That is,

$$\delta w \simeq \frac{1}{f(x,y)}\frac{\partial f}{\partial x}\delta x + \frac{1}{f(x,y)}\frac{\partial f}{\partial y}\delta y = \frac{1}{f(x,y)}\left[\frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y\right].$$

In other words,

$$\delta w \simeq \frac{1}{z}\left[\frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y\right].$$

Hence,

$$\delta w \simeq \frac{\delta z}{z}.$$

CONCLUSION

The fractional increment (or error) in z approximates to the actual increment (or error) in $\ln z$.

Multiplication by 100 will, of course, convert the fractional increment (or error) into a percentage.

Note:

The logarthmic method will apply equally well to a function of more than two independent variables where it takes the form of a product, a quotient, or a combination of these two.

EXAMPLES

1. If

$$w = \frac{x^3 z}{y^4},$$

   calculate, approximately, the percentage error in w when x is too small by 3%, y is too large by 1% and z is too large by 2%.

Solution

$$\ln w = 3\ln x + \ln z - 4\ln y,$$

giving

$$\frac{\delta w}{w} \simeq 3\frac{\delta x}{x} + \frac{\delta z}{z} - 4\frac{\delta y}{y},$$

where

$$\frac{\delta x}{x} = -\frac{3}{100}, \quad \frac{\delta y}{y} = \frac{1}{100} \text{ and } \frac{\delta z}{z} = \frac{2}{100}.$$

Hence,

$$\frac{\delta w}{w} \times 100 = -9 + 2 - 4 = -13.$$

Thus, w is too small by approximately 11%, as before.

2. In the formula

$$w = \sqrt{\frac{x^3}{y}},$$

x is subjected to an increase of 2%. Calculate, approximately, the percentage change needed in y to ensure that w remains unchanged.

Solution

$$\ln w = \frac{1}{2}[3\ln x - \ln y].$$

Hence,

$$\frac{\delta w}{w} \simeq \frac{1}{2}\left[3\frac{\delta x}{x} - \frac{\delta y}{y}\right],$$

where $\frac{\delta x}{x} = 0.02$
We require that $\delta w = 0$.
Thus,

$$0 = \frac{1}{2}\left[0.06 - \frac{\delta y}{y}\right],$$

giving

$$\frac{\delta y}{y} = 0.06$$

Hence, y must be approximately 6% too large.