"JUST THE MATHS"

SLIDES NUMBER

14.2

PARTIAL DIFFERENTIATION 2
(Partial derivatives of order higher than one)

by

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14.2.1 Standard notations and their meanings

UNIT 14.2
PARTIAL DIFFERENTIATION 2

PARTIAL DERIVATIVES OF ORDER HIGHER THAN ONE

14.2.1 STANDARD NOTATIONS AND THEIR MEANINGS

A partial derivative will, in general contain **all** of the independent variables.

We may need to differentiate again with respect to **any** of the independent variables.

If z is a function of two independent variables, x and y, the possible partial derivatives of the second order are

(i)

$$\frac{\partial^2 z}{\partial x^2}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right);$$

(ii)

$$\frac{\partial^2 z}{\partial y^2}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right);$$

(iii)

$$\frac{\partial^2 z}{\partial x \partial y}, \quad \text{which means} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right);$$

(iv)

$$\frac{\partial^2 z}{\partial y \partial x}, \quad \text{which means} \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right).$$

The last two can be shown to give the same result for all elementary functions likely to be encountered in science and engineering.

Note:

Occasionally, it may be necessary to use partial derivatives of order higher than two.

ILLUSTRATIONS

$$\frac{\partial^3 z}{\partial x \partial y^2} \quad \text{means} \quad \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]$$

and

$$\frac{\partial^4 z}{\partial x^2 \partial y^2} \quad \text{means} \quad \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right]\right).$$

EXAMPLES

Determine all the first and second order partial derivatives of the following functions:

1.

$$z = 7x^3 - 5x^2y + 6y^3.$$

Solution

$$\frac{\partial z}{\partial x} = 21x^2 - 10xy.$$

$$\frac{\partial z}{\partial y} = -5x^2 + 18y^2.$$

$$\frac{\partial^2 z}{\partial x^2} = 42x - 10y.$$

$$\frac{\partial^2 z}{\partial y^2} = 36y.$$

$$\frac{\partial^2 z}{\partial y \partial x} = -10x.$$

$$\frac{\partial^2 z}{\partial x \partial y} = -10x.$$

2.

$$z = y\sin x + x\cos y.$$

Solution

$$\frac{\partial z}{\partial x} = y\cos x + \cos y.$$

$$\frac{\partial z}{\partial y} = \sin x - x\sin y.$$

$$\frac{\partial^2 z}{\partial x^2} = -y\sin x.$$

$$\frac{\partial^2 z}{\partial y^2} = -x\cos y.$$

$$\frac{\partial^2 z}{\partial y\partial x} = \cos x - \sin y.$$

$$\frac{\partial^2 z}{\partial x\partial y} = \cos x - \sin y.$$

3.

$$z = e^{xy}(2x - y).$$

Solution

$$\frac{\partial z}{\partial x} = e^{xy}[y(2x - y) + 2] = e^{xy}[2xy - y^2 + 2].$$

$$\frac{\partial z}{\partial y} = e^{xy}[x(2x - y) - 1] = e^{xy}[2x^2 - xy - 1].$$

$$\frac{\partial^2 z}{\partial x^2} = e^{xy}[y(2xy - y^2 + 2) + 2y]$$

$$= e^{xy}[2xy^2 - y^3 + 4y].$$

$$\frac{\partial^2 z}{\partial y^2} = e^{xy}[x(2x^2 - xy - 1) - x]$$

$$= e^{xy}[2x^3 - x^2 y - 2x].$$

$$\frac{\partial^2 z}{\partial y \partial x} = e^{xy}[x(2xy - y^2 + 2) + 2x - 2y]$$

$$= e^{xy}[2x^2 y - xy^2 + 4x - 2y].$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{xy}[y(2x^2 - xy - 1) + 4x - y]$$

$$= e^{xy}[2x^2 y - xy^2 + 4x - 2y].$$